JUMPING NUMBERS OF F-PURE SUBMODULES

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ABSTRACT. The objective of this paper is to better understand F-injective thresholds and F-thresholds. Our approach is more general; we investigate the F-pure submodules of a module with a Cartier action, and relate their associated jumping numbers to numerical invariants of a Cartier algebra that resemble F-thresholds. As special cases, we obtain new results on the rationality and $\mathfrak{m}$-adic constancy of F-injective thresholds and F-thresholds.

1. INTRODUCTION

Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring of prime characteristic $p > 0$. Inspired by the test ideal introduced by Hochster and Huneke [HH90], Hara and Yoshida [HY03] defined a version associated to a pair which, in turn, prompted extensions to R-modules [Sch11, Bli13]. Given an R-module M, a Cartier algebra $\mathcal{C}$ on M, an ideal $\mathfrak{a}$ of R, and a real number $t \geqslant 0$, the *test module* of the triple $(M, \mathcal{C}, \mathfrak{a}^t)$ is a submodule of M that can be thought of as measuring the F-regularity of the triple [Bli13, Lemma 3.5].

The jumping numbers associated to test ideals and test modules have been the focus of intense research, with particular emphasis on their discreteness and rationality [BMS08, ST08, TT08, BMS09, KLZ09, BSTZ10], and on their $\mathfrak{m}$-adic constancy [HNnBWZ16, HNBW18, Sat21, Sat19]. When R is regular, the set of jumping numbers of the test ideals of the pair $\mathfrak{a}^t$, as t varies, coincide with the set of F-thresholds of $\mathfrak{a}$ [MTW05, BMS08]; this fact plays a key role in the proof of rationality of the jumping numbers of regular algebras of finite type over a field [BMS08], while also serving a fundamental tool in computations [Her14, Her15, BS15, GVJVNB]. However, the sets of jumping numbers and of F-thresholds differ in general (e.g., see [Hir09, MOY10, CM15]).

Our main focus herein is the study of *F-pure submodules* that, roughly speaking, measure the F-purity of a triple $(M, \mathcal{C}, \mathfrak{a}^t)$ (see Definition 3.1). Blickle introduced F-pure

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submodules in essentially the same way we present them here (please refer to Definition 2.14 and Proposition 2.17 for details on the comparison) [Bli13], while a slightly different notion for rings has also been studied under the name *non-F-pure ideals* [FST11, Sch14].

Our overarching goal is to understand the jumping numbers associated to F-pure submodules, with the ultimate aim of applying our results to F-thresholds and F-injective thresholds. Toward our broad goal, we define a family of thresholds associated to a Cartier algebra $\mathcal{C}$ over an R-module M, a submodule N of M, and an ideal $\mathfrak{a}$ of R (see Definition 3.3). Our first main result characterizes these Cartier thresholds, which we denote by $\operatorname{ct}^N_{\mathcal{C}}(\mathfrak{a})$, as the jumping numbers of F-pure submodules.

Theorem A (Theorem 3.6)**.** *Fix a finitely generated R-module M and let $\mathcal{C} = \bigoplus_e \mathcal{C}^e$ be a Cartier algebra on M such that $\mathcal{C}^1$ contains a surjective map. If $\mathfrak{a}$ is an ideal of R for which $\bigcup_e \mathcal{C}^e(\mathfrak{a}M) = M$, then the set*

$$\left\{\operatorname{ct}^N_{\mathcal{C}}(\mathfrak{a}) \mid N \subseteq M, \mathfrak{a} \subseteq \sqrt{\operatorname{ann}_R(M/N)}\right\}$$

coincides with the set of jumping numbers of the F-pure submodules of the triples $(M, \mathcal{C}, \mathfrak{a}^t)$, as $t \geqslant 0$ varies.

In Section 5, we realize certain well-studied numerical invariants as Cartier thresholds. Toward measuring the F-injectivity of a pair $(R, \mathfrak{a}^t)$ when R is F-rational, Schwede and Takagi defined the so-called *F-injective threshold* of $\mathfrak{a}$, which is rational when $\mathfrak{a}$ is principal [ST08, Remark 6.4 and Corollary 7.13]. Singh, Takagi, and Varbaro extended this class of numerical invariants to a larger family, each depending on an integer $i \geqslant 0$ [STV17]. The *i-th F-injective threshold* of $\mathfrak{a}$, $\operatorname{fit}_i(\mathfrak{a})$, is defined in terms of the Frobenius action on an i-th local cohomology module.

In Theorem 5.6, we prove that this larger family of F-injective thresholds are jumping numbers of F-pure submodules. Consequently, under certain assumptions, this family consists of rational numbers and satisfies $\mathfrak{m}$-adic constancy.

Theorem B (Corollary 5.7 and Theorem 5.12)**.** *Let $(R, \mathfrak{m})$ be an F-finite F-injective local ring of prime characteristic $p > 0$. If $H^i_{\mathfrak{m}}(R) \neq 0$, then the following hold.*

1. *If $\mathfrak{a}$ is an ideal of R not contained in any minimal prime, then $\operatorname{fit}_i(\mathfrak{a}) \in \mathbb{Q}$.*
2. *Given a parameter $f \in R$ such that R/fR is F-full and F-injective on the punctured spectrum, there exists an integer N for which $\operatorname{fit}_i(f) = \operatorname{fit}_i(f+h)$ whenever $h \in \mathfrak{m}^N$.*

Our results also apply to certain *F-thresholds* of pairs of ideals. Roughly speaking, an F-threshold is the limit of normalized Frobenius orders of an ideal with respect to another ideal (see Definition 3.1), and F-thresholds can be defined in any ring of prime characteristic [MTW05, HMTW08, DSNP18]. In Theorem 4.3, we show that for some singular rings, certain F-thresholds can be realized as Cartier thresholds, though in general, these two notions can differ. In analogy with Theorem B, we apply our more general theory to obtain results on the rationality and $\mathfrak{m}$-adic constancy of certain F-thresholds.

Theorem C (Corollary 5.15 and Theorem 5.16)**.** *Let $(R, \mathfrak{m})$ be a complete local F-finite Gorenstein domain of dimension d, and let J be an ideal of R generated by a full system of parameters $x_1, \ldots, x_d$. Then the following hold.*

1. *Given any ideal $\mathfrak{a}$ of R, $c^J(\mathfrak{a})$ is a rational number.*
2. *Given $f \in \mathfrak{m}$, there exists an integer N such that $c^J(f) = c^J(f+h)$ for every $h \in \mathfrak{m}^N$.*

2. Basic Properties of F-pure Submodules

Throughout, R denotes a commutative Noetherian ring of prime characteristic $p > 0$. We also assume that R is F-finite, i.e., the Frobenius map on R is finite. Given an integer $e \geqslant 0$, $\mathfrak{a}^{[p^e]}$ denotes the p^e-th Frobenius power of an ideal $\mathfrak{a}$ of R, the ideal generated by the p^e-th powers of the elements in $\mathfrak{a}$.

2.1. **Cartier algebras.** We briefly recall the notion of p^{-e}-linear maps and Cartier algebras, and we refer the reader to Blickle's work [Bli13] for more details.

Definition 2.1. Let M be a finitely generated R-module. A *p^{-e}-linear map on M* is an additive map $\varphi : M \to M$ such that $r\varphi(-) = \varphi(r^{p^e} \cdot -)$ for all $r \in R$. We denote by $\mathcal{C}^e_M = \operatorname{Hom}_e(M, M)$ the abelian group of p^{-e}-linear maps on M. Note that $\mathcal{C}_M = \bigoplus_{e\geqslant 0} \mathcal{C}^e_M$ is a ring, and also an R-algebra with structural map induced by viewing $r \in R$ as the "multiplication by r" map as an element of $\mathcal{C}^0_M = \operatorname{Hom}_R(M, M)$. Note that R is not, in general, central in $\mathcal{C}_M$. Given an element $\varphi \in \mathcal{C}^n_M$ for some $n > 0$, we let $R[\varphi] = \bigoplus_{e\geqslant 0} \varphi^e \cdot R$, where $\varphi^0 := \mathrm{id}_M$, and call it the *principal Cartier algebra generated by φ*.

More generally, a Cartier algebra is defined as follows.

Definition 2.2. A *Cartier algebra on R* is an $\mathbb{N}$-graded algebra $\mathcal{C} := \bigoplus_{e\geqslant 0} \mathcal{C}^e$ such that for each $r \in R$ and $\varphi \in \mathcal{C}^e$ one has $r \cdot \varphi = \varphi \cdot r^{p^e}$. As in Blickle's work [Bli13], we assume that the structural map $R \to \mathcal{C}^0$ is surjective.

If M is a finitely generated R-module, then a subalgebra $\mathcal{C} = \bigoplus_{e\geqslant 0} \mathcal{C}^e \subseteq \mathcal{C}_M$ is a Cartier algebra if $R \twoheadrightarrow \mathcal{C}^0$. Note that $R[\varphi]$ defined as above is always a Cartier algebra on R.

Throughout this paper, when dealing with a Cartier subalgebra $\mathcal{C} \subseteq \mathcal{C}_M$, we always assume that $\mathcal{C}^1$ contains a surjective element. In particular, when $R[\varphi]$ is a principal Cartier subalgebra of $\mathcal{C}_M$, we always assume that $\varphi \in \mathcal{C}^1_M$ is surjective.

Definition 2.3. We say that a Cartier subalgebra $\mathcal{C} = \bigoplus_{e\geqslant 0} \mathcal{C}^e$ of $\mathcal{C}_M$ is *principal* if there exists a surjective element $\varphi \in \mathcal{C}^1$ such that $\mathcal{C} = R[\varphi]$. If N is an R-submodule of M, we let $\mathcal{C}^e(N)$ denote the R-module generated by $\{\varphi(n) \mid n \in N, \varphi \in \mathcal{C}^e\}$. Finally, we let $\mathcal{C}(N) = \sum_e \mathcal{C}^e(N)$.

We record some basic properties of Cartier algebras.

Proposition 2.4. *Let $M \subseteq M'$ be R-modules, and $\mathcal{C}$ be a Cartier subalgebra of $\mathcal{C}_{M'}$.*

1. $\mathcal{C}^e(\mathfrak{a}M) \subseteq \mathcal{C}^{e+e'}(\mathfrak{a}^{[p^{e'}]}M)$ *for all* $e, e' \in \mathbb{N}$.
2. *If the Cartier algebra* $\mathcal{C}$ *is principal, then* $\mathcal{C}^e(\mathfrak{a}M) = \mathcal{C}^{e+e'}(\mathfrak{a}^{[p^{e'}]}M)$ *for every ideal* $\mathfrak{a}$ *of* R.
3. $\mathcal{C}^e(\mathfrak{a}^{\lceil tp^e \rceil}M) \subseteq \mathcal{C}^{e+1}(\mathfrak{a}^{\lceil tp^{e+1} \rceil}M)$ *for every integer* $e \geqslant 0$, *and real number* $t \geqslant 0$.

Proof. To see (1), let $\Phi \in \mathcal{C}^1$ be a surjective map, and let $x = \sum_i \varphi_i(y_i) \in \mathcal{C}^e(\mathfrak{a}M)$ for some $\varphi_i \in \mathcal{C}^e$ and $y_i \in \mathfrak{a}M$. Write $y_i = \sum_j a_{ij}m_{ij}$, with $a_{ij} \in \mathfrak{a}$ and $m_{ij} \in M$. Since Φ is surjective, we can find elements $m'_{ij} \in M$ such that $\Phi^{e'}(m'_{ij}) = m_{ij}$. For $z_i = \sum_j a_{ij}^{p^{e'}} m'_{ij} \in \mathfrak{a}^{[p^{e'}]}M$, we have that

$$x = \sum_i \varphi_i(y_i) = \sum_i \varphi_i \left(\sum_j a_{ij}\Phi^{e'}(m'_{ij}) \right) = \sum_i \varphi_i \left(\Phi^{e'}(z_i) \right) \in \mathcal{C}^{e+e'}(\mathfrak{a}^{[p^{e'}]}M).$$

For (2), if Φ is a surjective generator of $\mathcal{C}$, note that for any $x \in \mathfrak{a}$, we have $\Phi^{e+e'}(x^{p^{e'}}M) = \Phi^e(x\Phi^{e'}(M))$. Since $\Phi^{e'}(M) = M$, we obtain the equality. Finally, $\Phi^e(\mathfrak{a}M) = \mathcal{C}^e(\mathfrak{a}M)$ by our assumptions.

Finally, for part (3), note that $\mathcal{C}^e(\mathfrak{a}^{\lceil tp^e \rceil}M) \subseteq \mathcal{C}_M^{e+1}((\mathfrak{a}^{\lceil tp^e \rceil})^{[p]}M)$ by (2). Since $(\mathfrak{a}^{\lceil tp^e \rceil})^{[p]} \subseteq \mathfrak{a}^{p\lceil tp^e \rceil}$ and $p\lceil tp^e \rceil \geqslant \lceil tp^{e+1} \rceil$, this concludes the proof. □

Examples 2.5. The following are relevant examples of principal Cartier algebras:

1. The full Cartier algebra of a Gorenstein F-finite F-pure ring is principal.
2. If $(R, \mathfrak{m})$ is a complete Cohen-Macaulay F-finite F-injective ring of Krull dimension d, then the full Cartier algebra of the canonical module ω_R is principal. In this case, the Matlis dual of the Frobenius action on the top local cohomology module $H^d_{\mathfrak{m}}(R)$, the *Grothendieck trace of Frobenius*, is a generator.
3. If $(R, \mathfrak{m})$ is a complete F-finite F-injective ring, then we consider the principal Cartier algebra on the Matlis dual ω_i of $H^i_{\mathfrak{m}}(R)$ generated by the Matlis dual $F^\vee$ of the Frobenius map F on $H^i_{\mathfrak{m}}(R)$. Note that, since F is injective, its dual $F^\vee \in \mathcal{C}^1_{\omega_i}$ is surjective.

2.2. ***F*-pure submodules.** For completeness, we present some basic properties that F-pure submodules share with test ideals, which are well-known to experts. The techniques used are in large part analogous to those employed for test ideals [BMS08, KLZ09].

Definition 2.6. Let $\mathfrak{a}$ be an ideal of R, M a finitely generated R-module, and $\mathcal{C}$ a Cartier algebra on M. Given a real number $t \geqslant 0$, we define the *F-pure submodule* of M with respect to $\mathfrak{a}$ and t as

$$\sigma(M, \mathcal{C}, \mathfrak{a}^t) = \bigcup_{e \in \mathbb{N}} \mathcal{C}^e(\mathfrak{a}^{\lceil p^e t \rceil}M).$$

If $\mathfrak{a} = (f)$ is principal, we write $\sigma(M, \mathcal{C}, f^t)$ instead of $\sigma(M, \mathcal{C}, (f)^t)$.

By Proposition 2.4(3.), we have an increasing chain of submodules of M:

$$\cdots \subseteq \mathcal{C}^{e-1}(\mathfrak{a}^{\lceil p^{e-1}t \rceil}M) \subseteq \mathcal{C}^e(\mathfrak{a}^{\lceil p^e t \rceil}M) \subseteq \mathcal{C}^{e+1}(\mathfrak{a}^{\lceil p^{e+1}t \rceil}M) \subseteq \cdots \subseteq M.$$

Thus $\sigma(M,\mathcal{C},\mathfrak{a}^t)=\mathcal{C}^e(\mathfrak{a}^{\lceil p^e t\rceil}M)$ for all $e\gg 0$, because M is Noetherian.

The remainder of this section is devoted to demonstrating that F-pure submodules have similar features as test ideals.

Proposition 2.7. *Let $\mathfrak{a},\mathfrak{b}$ be ideals of R, M a finitely generated R-module and $\mathcal{C}$ a Cartier algebra on M. Given real numbers $s,t\geqslant 0$, the following properties hold.*

1. $\sigma(M,\mathcal{C},\mathfrak{a}^t)\subseteq\sigma(M,\mathcal{C},\mathfrak{a}^s)$ *if $s\leqslant t$, or if $\mathfrak{a}\subseteq\mathfrak{b}$.*
2. $\sigma(M,\mathcal{C},(\mathfrak{a}\cap\mathfrak{b})^t)\subseteq\sigma(M,\mathcal{C},\mathfrak{a}^t)\cap\sigma(M,\mathcal{C},\mathfrak{b}^t)$.
3. *If $\mathcal{C}$ is principal, then $\sigma(M,\mathcal{C},\mathfrak{a}^t)+\sigma(M,\mathcal{C},\mathfrak{b}^t)\subseteq\sigma(M,\mathcal{C},(\mathfrak{a}+\mathfrak{b})^t)$.*
4. *If W is a multiplicatively closed subset of R, then $W^{-1}\mathcal{C}=\mathcal{C}\otimes_R W^{-1}R$ is a Cartier algebra on $W^{-1}M$. What's more, $\sigma(W^{-1}M,W^{-1}\mathcal{C},W^{-1}\mathfrak{a}^t)\cong W^{-1}\sigma(M,\mathcal{C},\mathfrak{a}^t)$.*
5. *If R is local, then $\widehat{\mathcal{C}}=\mathcal{C}\otimes_R\widehat{R}$ is a Cartier algebra on $\widehat{M}$, and*
$$\sigma(\widehat{M},\widehat{\mathcal{C}},\widehat{\mathfrak{a}}^t)\cong\sigma(M,\mathcal{C},\mathfrak{a}^t)\otimes_R\widehat{R}.$$
6. *If $\mathcal{C}$ is principal, then $\sigma(M,\mathcal{C},f^{r/p^e})=\mathcal{C}^e(f^rM)$ for all $f\in R$ and $e,r\in\mathbb{N}$.*

Proof. As the claimed properties are well-known and the proofs are similar to those for test ideals [BMS08, KLZ09], we give only a sketch of the arguments.

To see (1) when $s\geqslant t$, observe that since $\mathfrak{a}^{\lceil tp^e\rceil}\subseteq\mathfrak{a}^{\lceil sp^e\rceil}$ for all $e\geqslant 0$, we have that $\mathcal{C}^e(\mathfrak{a}^{\lceil tp^e\rceil})\subseteq\mathcal{C}^e(\mathfrak{a}^{\lceil sp^e\rceil})$. When $\mathfrak{a}\subseteq\mathfrak{b}$, the inclusion follows from the fact that $\mathfrak{a}^{\lceil tp^e\rceil}\subseteq\mathfrak{b}^{\lceil tp^e\rceil}$ for any $e\geqslant 0$ since $\mathfrak{a}\subseteq\mathfrak{b}$, and (2) follows directly from (1) when $\mathfrak{a}\subseteq\mathfrak{b}$. For (3), assume that $\Phi\in\mathcal{C}^1$ is a generator of $\mathcal{C}$, and observe that
$$\Phi^e(\mathfrak{a}^{\lceil tp^e\rceil}M)+\Phi^e(\mathfrak{b}^{\lceil tp^e\rceil}M)=\Phi^e(\mathfrak{a}^{\lceil tp^e\rceil}M+\mathfrak{b}^{\lceil tp^e\rceil}M)\subseteq\Phi^e((\mathfrak{a}+\mathfrak{b})^{\lceil tp^e\rceil}M).$$
The claim follows since $\sigma(M,\mathcal{C},\mathfrak{a}^t)=\Phi^e(\mathfrak{a}^{\lceil tp^e\rceil})$ and $\sigma(M,\mathcal{C},\mathfrak{b}^t)=\Phi^e(\mathfrak{b}^{\lceil tp^e\rceil})$ for $e\gg 0$. The statement in (4) holds because $W^{-1}\varphi(\mathfrak{a}^{\lceil tp^e\rceil}M)=\varphi((W^{-1}\mathfrak{a})^{\lceil tp^e\rceil}M)$ for all $e\geqslant 0$ and all $\varphi\in\mathcal{C}^e$, so that $\sigma(M,\mathcal{C},\mathfrak{a}^t)=\mathcal{C}^e(\mathfrak{a}^{\lceil tp^e\rceil})$ for $e\gg 0$. The proof of (5) is analogous to that of (4). For (6), let $\Phi\in\mathcal{C}^1$ be a generator of $\mathcal{C}$. We have that for $e'\gg 0$,
$$\sigma(M,\mathcal{C},f^{r/p^e})=\mathcal{C}^{e+e'}(f^{rp^{e'}})=\Phi^{e+e'}(f^{rp^{e'}}M).$$
Since $\Phi^{e+e'}(f^{rp^{e'}}M)=\Phi^e(f^rM)$, we obtain the desired conclusion. ◻

Proposition 2.8. *Let M be a finitely generated R-module and $\mathcal{C}$ a Cartier algebra on M. Assume that $\mathfrak{a}$ is an ideal of R for which $\mathcal{C}(\mathfrak{a}M)=M$. Then for each $t>0$, there exists $\varepsilon>0$ such that $\sigma(M,\mathcal{C},\mathfrak{a}^t)=\sigma(M,\mathcal{C},\mathfrak{a}^s)$ for all $s\in[t,t+\varepsilon)$. Additionally, $\sigma(M,\mathcal{C},\mathfrak{a}^t)=\mathcal{C}^e(\mathfrak{a}^r)$ for all $r,e\in\mathbb{N}$ such that $\frac{r}{p^e}\in(t,t+\varepsilon)$.*

Proof. The proof is analogous to the proof of the corresponding statement for test ideals [BMS08, Proposition 2.14]. ◻

Proposition 2.8 allows us to define jumping numbers in our context.

Definition 2.9. Let $\mathfrak{a}$ be an ideal of R, M a finitely generated R-module, and $\mathcal{C}$ a Cartier algebra on M. We say that a real number $t>0$ is a *σ-jumping number* of $\sigma(M,\mathcal{C},\mathfrak{a}^\bullet)$ if $\sigma(M,\mathcal{C},\mathfrak{a}^t)\neq\sigma(M,\mathcal{C},\mathfrak{a}^{t-\varepsilon})$ for all $\varepsilon>0$.

In Section 3, we give an interpretation of the σ-jumping numbers in terms of numerical invariants that are intimately related to F-thresholds and F-pure thresholds.

Smirnov and Tucker introduced the notion of Cartier signature as a modification of the F-signature [ST19]. This number is defined in terms of the trace map on ω_R, and thus on its Cartier structure. In a similar spirit, we introduce a threshold that takes into account Cartier structures on an arbitrary R-module, not just on ω_R.

Definition 2.10. Let $\mathfrak{a}$ be an ideal of R, M an R-module, and $\mathcal{C}$ a Cartier algebra on M. We define the *Cartier-pure threshold* of $\mathfrak{a}$ in M with respect to $\mathcal{C}$ as

$$\operatorname{cpt}(M,\mathcal{C},\mathfrak{a}) = \inf\{t \in \mathbb{R}_{\geqslant 0} \mid \sigma(M,\mathcal{C},\mathfrak{a}^t) \neq M\}.$$

We now seek to compare the modules $\sigma(M,\mathcal{C},\mathfrak{a}^t)$ to Blickle's version of F-pure submodules $\underline{M}_{\mathcal{C}^{\mathfrak{a}^t}}$ [Bli13]. For this, we recall the following definition.

Definition 2.11 ([Bli13, Definition 3.4])**.** Let $\mathcal{C}$ be a Cartier algebra on an R-module M. We say that M *is F-regular with respect to* $\mathcal{C}$ if there is no proper non-zero $\mathcal{C}$-submodule $N \subseteq M$ that generically agrees with M.

The following is a slight modification of the definition of test ideal given by Schwede [Sch11, Definition 3.16], where we replace the value $\lceil tp^e \rceil$ appearing in his definition with $\lceil t(p^e-1)\rceil$. Note that R° denotes the elements of a Noetherian ring R that are contained in no minimal prime of R.

Definition 2.12. Fix an ideal $\mathfrak{a}$ of a Noetherian ring R of characteristic $p > 0$, and a Cartier algebra $\mathcal{C}$ on R. Given a real number $t \geqslant 0$, $\tau_b(R,\mathcal{C},\mathfrak{a}^t)$ is the smallest nonzero ideal J such that $J \cap R^\circ \neq \varnothing$ and for all $e \geqslant 0$, $\varphi(\mathfrak{a}^{\lceil tp^e \rceil}J) \subseteq J$ for all $\varphi \in \mathcal{C}^e$.

Let R be a complete F-finite F-pure local domain that is not strongly F-regular, and $\mathcal{C}$ be the complete Cartier algebra of R. Then $\sigma(R,\mathcal{C},R^t) = R$ for every t, but $\tau_b(R,\mathcal{C},R^t) = \tau_b(R) \neq R$, where $\tau_b(R)$ is the big test ideal [HH89] (see also [ST12]).

The ideal $\tau_b(R,\mathfrak{a}^t,\mathcal{C}_R)$ coincides with Blickle's notion of test ideals that are defined as the stable value of a certain decreasing chain of ideals [Bli13, Section 3.3]. Blickle introduced a more general version of test ideals, called *test modules*, associated to a Cartier algebra $\mathcal{C}$ on a R-module M, which exist under mild assumptions on R and M [Bli13]. If $\mathcal{C}$ is finitely generated, then the jumping numbers of the test modules $\tau(M,\mathcal{C},\mathfrak{a}^\bullet)$ are discrete [Bli13, Section 4.3].

Definition 2.13 ([Bli13, Definition 3.1])**.** Fix an ideal $\mathfrak{a}$ of a Noetherian ring R of prime characteristic $p > 0$, and a real number $t \geqslant 0$. Let M be a finitely generated R-module, and $\mathcal{C}$ a Cartier algebra on M. The test module $\tau(M,\mathcal{C},\mathfrak{a}^t)$ is defined as the smallest submodule N of M such that

1. $\bigcap_{e\geqslant 0} \mathcal{C}^e(\mathfrak{a}^{\lceil p^e \rceil}N) \subseteq N$, and
2. $N_\eta = \sigma(M,\mathcal{C},\mathfrak{a}^t)_\eta$ for every generic point η of a component of $\operatorname{Supp}_R(\sigma(M,\mathcal{C},\mathfrak{a}^t))$.

We now recall a characterization of test modules based on Schwede's proof of the existense of test ideals, adapted to test modules.

Theorem 2.14 ([Sch11, Proofs of Theorem 3.18 and Proposition 3.21])**.** *Let $\mathcal{C}$ be a Cartier algebra on an R-module M, and take $f \in R^0$ such that M_f is strongly F-regular with respect to $\mathcal{C}_f$. Let $\mathfrak{a}$ be an ideal of R, and let $t \geqslant 0$ be a real number. There exist $m \in \mathbb{Z}_{>0}$ such that*

$$\tau(M, \mathcal{C}, \mathfrak{a}^t) = \sum_{e \geqslant 0} \mathcal{C}^e \mathfrak{a}^{\lceil t(p^e-1) \rceil} f^m M.$$

Furthermore, if $m' > m$, then

$$\tau(M, \mathcal{C}, \mathfrak{a}^t) = \sum_{e \geqslant 0} \mathcal{C}^e \mathfrak{a}^{\lceil t(p^e-1) \rceil} f^{m'} M.$$

Remark 2.15. Suppose that R is a domain. Let $\mathcal{C}$ be a Cartier algebra on an R-module M such that $(M, \mathcal{C})$ is F-regular. Let $\mathfrak{a} \neq 0$ be an ideal of R, and let $t \geqslant 0$ be a real number. Then M_f is strongly F-regular with respect to $\mathcal{C}_f$ for every $f \in \mathfrak{a}$. Thus

$$\tau(M, \mathcal{C}, \mathfrak{a}^t) = \sum_{e \geqslant 0} [\mathcal{C}^{\mathfrak{a}^t}]^e \mathfrak{a}^m M = \sum_{e \geqslant 0} \mathcal{C}^e \mathfrak{a}^{\lceil t(p^e-1) \rceil} \mathfrak{a}^m M$$

for some $m \in \mathbb{Z}$.

The following notation, analogous to standard notation in different contexts, is adopted for the remainder of this article.

Notation 2.16. *Let $\mathfrak{a}$ be an ideal of R, M a finitely generated R-module, and $\mathcal{C}$ a Cartier algebra on M. For a given real number $t > 0$, we let*

$$\sigma(M, \mathcal{C}, \mathfrak{a}^{t-\varepsilon}) = \bigcap_{\lambda < t} \sigma(M, \mathcal{C}, \mathfrak{a}^{\lambda}).$$

In a similar fashion, we let $\underline{M}_{\mathcal{C}^{\mathfrak{a}^{t-\varepsilon}}} = \bigcap_{\lambda < t} \underline{M}_{\mathcal{C}^{\mathfrak{a}^{\lambda}}}$, and $\underline{M}_{\mathcal{C}^{\mathfrak{a}^{t+\varepsilon}}} = \bigcup_{\lambda > t} \underline{M}_{\mathcal{C}^{\mathfrak{a}^{\lambda}}}$.

In the following proposition we make precise the comparison between our F-pure submodules, and the ones defined by Blickle, by showing that $\sigma(M, \mathcal{C}, \mathfrak{a}^t) = \underline{M}_{\mathcal{C}^{\mathfrak{a}^{t+\varepsilon}}}$. Experts might recognize the difference in how t is approximated as the feature distinguishing F-purity from sharp F-purity [Her12, Theorem 0.2].

Proposition 2.17. *Suppose that R is a domain. Let $\mathcal{C}$ be a Cartier algebra on an R-module M such that $(M, \mathcal{C})$ is F-regular, and $\mathfrak{a}$ is an ideal of R. Given a real number $t > 0$, there exists $\varepsilon > 0$ such that $\sigma(M, \mathcal{C}, \mathfrak{a}^t) = \underline{M}_{\mathcal{C}^{\mathfrak{a}^s}}$ whenever $s \in (t, t+\varepsilon)$.*

Proof. By Proposition 2.8, there exists $\varepsilon > 0$ such that $\sigma(M, \mathcal{C}, \mathfrak{a}^t) = \mathcal{C}^e(\mathfrak{a}^r M)$ for all integers $r, e > 0$ such that $t < r/p^e < t + \varepsilon$, and $\sigma(M, \mathcal{C}, \mathfrak{a}^t) = \sigma(M, \mathcal{C}, \mathfrak{a}^s)$ for every $s \in (t, t+\varepsilon)$. We fix $s \in (t, t+\varepsilon)$ and take an integer $m > s$ be such that

$$\tau(M, \mathcal{C}, \mathfrak{a}^s) = \sum_{e \geqslant 0} \mathcal{C}^e \mathfrak{a}^{\lceil s(p^e-1) \rceil} \mathfrak{a}^m M,$$

which is possible by Remark 2.15. Since the sequence $\left\{ \frac{\lceil s(p^e-1) \rceil}{p^e} \right\}$ is nonincreasing and converges to s, we have that $t < \frac{\lceil s(p^e-1) \rceil + m}{p^e} < t + \varepsilon$ for $e \gg 0$. Then, $\sigma(M, \mathcal{C}, \mathfrak{a}^t) = \mathcal{C}^e(\mathfrak{a}^{\lceil s(p^e-1) \rceil + m} M)$ for $e \gg 0$. We conclude that $\sigma(M, \mathcal{C}, \mathfrak{a}^t) = \bigcup_{e \geqslant 0} \mathcal{C}^e(\mathfrak{a}^{\lceil s(p^e-1) \rceil + m} M) = \underline{M}_{\mathcal{C}^{\mathfrak{a}^s}}$. ☐

There are examples where $\sigma(M, \mathcal{C}, \mathfrak{a}^t)$ and $\underline{M}_{\mathcal{C}^{\mathfrak{a}^t}}$ coincide, and examples where they differ: Let $R = \mathbb{F}_2[x]$, and let $\mathcal{C}$ be the full Cartier algebra on R. If $\mathfrak{a} = (x^{p-1})$ and $t = \frac{1}{p-1}$, then $\sigma(M, \mathcal{C}, \mathfrak{a}^t) = (x) = \underline{M}_{\mathcal{C}^{\mathfrak{a}^t}}$. In contrast, if $\mathfrak{a} = (x^p)$ and $t = \frac{1}{p-1}$, then $\sigma(M, \mathcal{C}, \mathfrak{a}^t) = R \neq (x) = \underline{M}_{\mathcal{C}^{\mathfrak{a}^t}}$.

Even if the two modules differ, though, they still have the same jumping numbers.

Corollary 2.18. *Let $\mathcal{C}$ be a Cartier algebra on an R-module M, $\mathfrak{a}$ is an ideal of R, and $t \geqslant 0$ a real parameter. Then t is a jumping number for $\sigma(M, \mathcal{C}, \mathfrak{a}^\bullet)$ if and only if t is a jumping number for $\underline{M}_{\mathcal{C}^{\mathfrak{a}^\bullet}}$.* □

Lemma 2.19. *Let M be a finitely generated R-module, and $\mathfrak{a}$ an ideal of R. If $\mathcal{C}$ is a Cartier algebra on M, then the following conditions are equivalent.*

1. $\operatorname{cpt}(M, \mathcal{C}, \mathfrak{a}) > 0$.
2. $\sigma(M, \mathcal{C}, \mathfrak{a}^t) = M$ *for some* $t > 0$.
3. $\mathcal{C}(\mathfrak{a}M) = M$.

Proof. The equivalence of (1) and (2) follows from the definition of jumping numbers because $\sigma(M, \mathcal{C}, \mathfrak{a}^0) = M$. To see that (2) implies (3) note that $M = \mathcal{C}^e(\mathfrak{a}^{\lceil tp^e \rceil} M)$ for $e \gg 0$. Choose $e \gg 0$ such that $tp^e \geqslant 1$, so that $\mathfrak{a}^{\lceil tp^e \rceil} \subseteq \mathfrak{a}$. Then $M = \mathcal{C}^e(\mathfrak{a}^{\lceil tp^e \rceil} M) \subseteq \mathcal{C}^e(\mathfrak{a}M) \subseteq M$. Finally, for (3) implies (2), fix e such that $\mathcal{C}^e(\mathfrak{a}M) = M$. Observe that $M = \mathcal{C}^e(\mathfrak{a}M) = \mathcal{C}^e(\mathfrak{a}^{\lceil \frac{1}{p^e} p^e \rceil} M) \subseteq \mathcal{C}_M^{e+e'}(\mathfrak{a}^{\lceil \frac{1}{p^e} p^{e+e'} \rceil} M) \subseteq M$ for all $e' \geqslant 0$, so that $\sigma(\mathfrak{a}^{\frac{1}{p^e}}) = M$. □

We now state a Skoda-type result for F-pure submodules, and a statement that the set of jumping coefficients is closed. The corresponding properties are essential features of test ideals. We omit the proofs of these results, since they are analogous to the corresponding ones for test ideals.

Proposition 2.20 ([BMS08, Proposition 2.25])**.** *If $\mathcal{C}$ is a Cartier algebra on an R-module M, and an ideal $\mathfrak{a}$ of R is generated by s elements, then for all $t > s$ we have*

$$\sigma(M, \mathcal{C}, \mathfrak{a}^t) = \mathfrak{a}\, \sigma(M, \mathcal{C}, \mathfrak{a}^{t-1}).$$

Proposition 2.21 ([KLZ09, Lemma 2.4])**.** *Let $\mathfrak{a}$ be an ideal of R, M a finitely generated R-module, and $\mathcal{C}$ be a Cartier algebra on M such that $\mathcal{C}(\mathfrak{a}M) = M$. Then the set of all σ-jumping numbers of $\sigma(M, \mathcal{C}, \mathfrak{a}^\bullet)$ is closed.*

2.3. **Discreteness of jumping numbers of F-pure submodules.** In this subsection, we identify settings in which the set of σ-jumping numbers is discrete.

Theorem 2.22 ([Bli13, Proposition 4.9 and Theorem 4.18])**.** *Let R be an F-finite ring, M a finitely generated R-module, and $\mathcal{C}$ a Cartier algebra on M. If $\mathcal{C}$ is finitely generated as an R-algebra, then the set of jumping numbers of $\sigma(M, \mathcal{C}, \mathfrak{a}^\bullet)$ has no accumulation points.*

Proof. From Proposition 2.17, the proof follows directly from [Bli13, Proposition 4.9] and [Bli13, Theorem 4.18]. □

Proposition 2.23. *Let M be a finitely generated R-module, and let $\mathcal{C}$ be a Cartier algebra on M. Suppose that $\mathfrak{a}$ is an ideal of R such that $\operatorname{cpt}(M, \mathcal{C}, \mathfrak{a}) > 0$ and $\operatorname{Supp}_R(M/\mathfrak{a}M)$ has dimension zero. Then the set of jumping numbers of $\sigma(M, \mathcal{C}, \mathfrak{a}^\bullet)$ has no accumulation points.*

Proof. By way of contradiction, suppose that $t \in \mathbb{R}_{>0}$ is a accumulation point of jumping numbers, so that t is also a jumping number by Proposition 2.21. Fix a sequence $\{t_n\}$ converging to t, which we can assume is increasing by By Proposition 2.8.

For nonnegative integers n, let $M_n = \sigma(M, \mathcal{C}, \mathfrak{a}^{t_n})$, so that $\{M_n\}_{n=0}^{\infty}$ is a descending chain of modules. Set $N = \bigcap_{n\geqslant 0} M_n$. Let $r \in \mathbb{N}$ be such that $m > t$. We observe that

$$\mathfrak{a}^r M = \mathfrak{a}^r \mathcal{C}^e M = \mathcal{C}^e\left((\mathfrak{a}^m)^{[p^e]} M\right) \subseteq \mathcal{C}^e\left((\mathfrak{a}^{mp^e} M\right) \subseteq \sigma(M, \mathcal{C}, \mathfrak{a}^m) \subseteq M_n$$

for every n. Then, $\mathfrak{a}^r M \subseteq N$.

We note that $\operatorname{Supp}(M/\mathfrak{a}^r M) = \operatorname{Supp}(M/\mathfrak{a}M)$. Since $\operatorname{Supp}(M/\mathfrak{a}M) \subseteq \operatorname{Supp}(M/N) \subseteq \operatorname{Supp}(M/\mathfrak{a}^r M)$, we have that $\operatorname{Supp}(M/\mathfrak{a}M) = \operatorname{Supp}(M/N)$. Then, M/N has finite length, and so, the sequence $\{M_n\}$ is eventually constant, or, equivalently, that $M_n = N$ for all $n \gg 0$. ☐

3. Cartier Thresholds as σ-jumping Numbers

We continue to adopt the notation established in Section 2. Before building our setup, we recall the definition of an F-threshold.

Definition 3.1 ([MTW05, HMTW08, DSNP18])**.** Given proper ideals $\mathfrak{a}, J$ of R for which $\mathfrak{a} \subseteq J$, and an integer $e \geqslant 0$, let $\nu_{\mathfrak{a}}^J(p^e) = \max\{t \in \mathbb{N} \mid \mathfrak{a}^t \not\subseteq J^{[p^e]}\}$. The limit

$$c^J(\mathfrak{a}) = \lim_{e\to\infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}$$

exists, and it is called the *F-threshold* of $\mathfrak{a}$ with respect to J.

Now consider a Cartier algebra $\mathcal{C} = \bigoplus_e \mathcal{C}^e$ on a finitely generated R-module M, and any submodule N of M. For a fixed integer $e \geqslant 0$, we define

$$I_e(N, \mathcal{C}) = \{x \in M \mid \varphi(x) \in N \text{ for all } \varphi \in \mathcal{C}^e\}.$$

If the Cartier algebra $\mathcal{C}$ is clear from the context, we often write $I_e(N)$ for $I_e(N, \mathcal{C})$. Observe that $J^{[p^e]}M \subseteq I_e(JM)$ for any ideal J and any R-module M.

Fix finitely generated modules $N \subseteq M$, and suppose that an ideal $\mathfrak{a}$ is contained in $\sqrt{J}$, where $J = \operatorname{ann}_R(M/N)$. For all integers $e \geqslant 0$, there exists an integer t_e such that $\mathfrak{a}^{t_e} \subseteq J^{[p^e]}$. for all $\varphi \in \mathcal{C}^e$, we have

$$\varphi(\mathfrak{a}^{t_e} M) \subseteq \varphi(J^{[p^e]} M) = J\varphi(M) \subseteq JM \subseteq N.$$

Therefore, the following integer is well-defined:

$$\delta_{\mathfrak{a}}^{N,\mathcal{C}}(p^e) = \max\{t \in \mathbb{N} \mid \mathfrak{a}^t M \not\subseteq I_e(N, \mathcal{C})\}.$$

Observe that the dependence on M, even if reflected by the presence of $\mathcal{C}$, is omitted from the notation in order to avoid making it clunkier.

The following preparatory lemma result permits us to define an analog of F-thresholds in our context.

Lemma 3.2. *Let $N \subseteq M$ be finitely generated R-modules, and $\mathcal{C}$ be a Cartier algebra on M. Assume that there exists a surjective map in $\mathcal{C}^1$. Let $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a} \subseteq \sqrt{J}$, where $J = \operatorname{ann}_R(M/N)$. Then*

$$\frac{\delta_{\mathfrak{a}}^{N,\mathcal{C}}(p^e)}{p^e} \leqslant \frac{\delta_{\mathfrak{a}}^{N,\mathcal{C}}(p^{e+1})}{p^{e+1}} \leqslant \frac{\nu_{\mathfrak{a}}^{J}(p^{e+1})}{p^{e+1}}$$

for all integers $e \geqslant 0$.

Proof. Let $t = \delta_{\mathfrak{a}}^{N,\mathcal{C}}(p^e)$. By definition, there exists $f \in \mathfrak{a}^t$, an element $m \in M$, and a map $\varphi \in \mathcal{C}^e$ such that $\varphi(fm) \notin N$. If $\Phi \in \mathcal{C}^1$ is a surjective map, then $\psi := \varphi \circ \Phi \in \mathcal{C}^{e+1}$. Let $u \in M$ be such that $\Phi(u) = m$. Since Φ is p^{-1}-linear, we have

$$\psi\left(f^p u\right) = \varphi\left(f\Phi(u)\right) = \varphi\left(fm\right) \notin N.$$

Since $f^p u \in \mathfrak{a}^{tp}M$, we have that $\mathfrak{a}^{tp}M \not\subseteq I_{e+1}(N,\mathcal{C})$, and so $\delta_{\mathfrak{a}}^{N,\mathcal{C}}(p^{e+1}) \geqslant tp$; dividing this equation by p^{e+1} gives the first inequality. For the other inequality, observe that if $s = \nu_{\mathfrak{a}}^{J}(p^{e+1}) + 1$, then $\mathfrak{a}^s M \subseteq J^{[p^e]}M \subseteq I_e(N,\mathcal{C})$. Hence $\varphi(\mathfrak{a}^s M) \subseteq N$ for all $\varphi \in \mathcal{C}_M^{e+1}$. Hence $s \geqslant \delta_{\mathfrak{a}}^{N,\mathcal{C}}(p^{e+1}) + 1$, giving the second inequality after dividing by p^{e+1}. □

Definition 3.3. Under the assumptions of Lemma 3.2, we define the *Cartier-threshold* of $(M, \mathcal{C}, \mathfrak{a})$ with respect to N as

$$\operatorname{ct}_{\mathcal{C}}^{N}(\mathfrak{a}) = \lim_{e \to \infty} \frac{\delta_{\mathfrak{a}}^{N,\mathcal{C}}(p^e)}{p^e} = \sup_{e \geqslant 1} \frac{\delta_{\mathfrak{a}}^{N,\mathcal{C}}(p^e)}{p^e}$$

which is guaranteed to exist.

If $J = \operatorname{ann}_R(M/N)$, then $\operatorname{ct}_{\mathcal{C}}^{N}(\mathfrak{a})$ is bounded above by the F-threshold $c^J(\mathfrak{a})$ by Lemma 3.2. In fact, the Cartier threshold can be approximated by F-thresholds.

Theorem 3.4. *Let $N \subseteq M$ be finitely generated R-modules, and $\mathcal{C}$ be a Cartier subalgebra of $\mathcal{C}_M$. Furthermore, assume that either $\mathcal{C}$ is principal, or that $M = R$ and $\mathcal{C} = \mathcal{C}_R$. Let $\mathfrak{a}$ be an ideal of R contained in $\sqrt{J}$, where $J = \operatorname{ann}_R(M/N)$. If $J_e = \operatorname{ann}_R(M/I_e(N,\mathcal{C}))$, then*

$$\operatorname{ct}_{\mathcal{C}}^{N}(\mathfrak{a}) = \lim_{e \to \infty} \frac{c^{J_e}(\mathfrak{a})}{p^e}.$$

Proof. For all integers $e, e' \geqslant 0$, we know that

$$0 \leqslant \frac{\nu_{\mathfrak{a}}^{J_e}(p^{e'})}{p^{e'}} - \nu_{\mathfrak{a}}^{J_e}(p^0) \leqslant \mu(\mathfrak{a}). \tag{3.0.1}$$

where $\mu(P)$ denotes the minimal number of generators of an R-module P [DSNP18, Lemma 3.3]. We observe that $\nu_{\mathfrak{a}}^{J_e}(p^0) = \delta_{\mathfrak{a}}^{N,\mathcal{C}}(p^e)$. In fact, if $t = \nu_{\mathfrak{a}}^{J_e}(p^0)$, then $\mathfrak{a}^{t+1} \subseteq J_e$, so that $\mathfrak{a}^{t+1}M \subseteq J_e M \subseteq I_e(N,\mathcal{C})$. This implies that $t \geqslant \delta_{\mathfrak{a}}^{N,\mathcal{C}}(p^e)$. Conversely, we know that $\mathfrak{a}^t \not\subseteq J_e$. Hence there exists $f \in \mathfrak{a}^t$ and $m \in M$ such that $fm \notin I_e(N,\mathcal{C})$. As a consequence, we have that $\mathfrak{a}^t M \not\subseteq I_e(N,M)$, and this shows that $t = \nu_{\mathfrak{a}}^{J_e}(p^0) = \delta_{\mathfrak{a}}^{N,\mathcal{C}}(p^e)$. Now, taking the limit as $e' \to \infty$ in (3.0.1), and dividing by p^e for $e \in \mathbb{N}$, we obtain

$$0 \leqslant \frac{c^{J_e}(\mathfrak{a})}{p^e} - \frac{\delta_{\mathfrak{a}}^{N,\mathcal{C}}(p^e)}{p^e} \leqslant \frac{\mu(\mathfrak{a})}{p^e}$$

Given that $\mu(\mathfrak{a})/p^e \to 0$ as $e \to \infty$, this completes the proof. □

We now observe that when $M = R$ and $\mathcal{C}$ is the full Cartier algebra on R, F-pure thresholds can be realized as Cartier thresholds. In what follows, $\operatorname{fpt}(\mathfrak{a})$ denotes the F-pure threshold of $\mathfrak{a}$.

Corollary 3.5. *Let $(R,\mathfrak{m},K)$ be either a local ring, or a standard graded k-algebra, that is F-finite and F-pure. Let $\mathcal{C}_R = \bigoplus_e \operatorname{Hom}_e(R,R)$ be the full Cartier algebra on R. Then for an ideal $\mathfrak{a}$ of R, which we assume is homogeneous in the case that R is graded,*

$$\operatorname{ct}_{\mathcal{C}_R}^{\mathfrak{m}}(\mathfrak{a}) = \operatorname{fpt}(\mathfrak{a}).$$

Proof. Since R is F-pure, $N = \mathfrak{m}$, $M = R$, and $\mathcal{C} = \mathcal{C}_R$ the full Cartier algebra on R satisfy the assumptions of Theorem 3.4, so that $\operatorname{ct}_{\mathcal{C}_R}^{\mathfrak{m}}(\mathfrak{a}) = \lim_{e\to\infty} \frac{c^{J_e}(\mathfrak{a})}{p^e}$. Since $J_e = I_e(\mathfrak{m})$, we can conclude that $\operatorname{ct}_{\mathcal{C}_R}^{\mathfrak{m}}(\mathfrak{a}) = \operatorname{fpt}(\mathfrak{a})$ [DSNP18, Theorem 4.6]. □

Our next result is analogous the fact that the set of F-jumping numbers in a regular ring equals the set of F-thresholds [BMS08, Corollary 2.30].

Theorem 3.6. *Let M be a finitely generated R-module, and let $\mathcal{C}$ be a Cartier algebra on M. Suppose that $\mathfrak{a}$ is an ideal of R for which $\mathcal{C}(\mathfrak{a}M) = M$. Then the set $\{\operatorname{ct}_{\mathcal{C}}^N(\mathfrak{a}) \mid N \subseteq M, \mathfrak{a} \subseteq \sqrt{\operatorname{ann}_R(M/N)}\}$ coincides with the set of jumping numbers of the F-pure submodules $\{\sigma(M,\mathcal{C},\mathfrak{a}^t) \mid t \in \mathbb{R}_{\geqslant 0}\}$.*

Proof. First fix a submodule N of M for which $\mathfrak{a} \subseteq \sqrt{\operatorname{ann}_R(M/N)}$, and let $\alpha = \operatorname{ct}_{\mathcal{C}}^N(\mathfrak{a})$. By Proposition 2.8, there exists $s > \alpha$ such that $\sigma(M,\mathcal{C},\mathfrak{a}^\alpha) = \sigma(M,\mathcal{C},\mathfrak{a}^s)$. Since $s > \alpha \geqslant \delta_{\mathfrak{a}}^{N,\mathcal{C}}(p^e)/p^e$ for all $e \geqslant 0$, we conclude that $\lceil sp^e \rceil \geqslant \delta_{\mathfrak{a}}^{N,\mathcal{C}}(p^e) + 1$ for all $e \geqslant 0$. Therefore, $\mathcal{C}^e(\mathfrak{a}^{\lceil sp^e \rceil} M) \subseteq N$ for all $e \geqslant 0$, and thus $\sigma(M,\mathcal{C},\mathfrak{a}^\alpha) = \sigma(M,\mathcal{C},\mathfrak{a}^s) \subseteq N$. Fixing$\varepsilon > 0$, for $e \gg 0$, we have that $(\alpha - \varepsilon)p^e \leqslant \delta_{\mathfrak{a}}^{N,\mathcal{C}}(p^e)$. Hence $\mathfrak{a}^{\lceil (\alpha-\varepsilon)p^e \rceil} M \not\subseteq I_e(N,\mathcal{C})$ for all $e \gg 0$, which means that $\sigma(M,\mathcal{C},\mathfrak{a}^{\alpha-\varepsilon}) \not\subseteq N$. In particular, $\sigma(M,\mathcal{C},\mathfrak{a}^\alpha) \subsetneq \sigma(M,\mathcal{C},\mathfrak{a}^{\alpha-\varepsilon})$. Since this holds for each $\varepsilon > 0$, $\alpha = \operatorname{ct}_{\mathcal{C}}^N(\mathfrak{a})$ is a jumping number of $\sigma(M,\mathcal{C},\mathfrak{a}^\bullet)$.

Conversely, assume that t is a jumping number so that $\sigma(M,\mathcal{C},\mathfrak{a}^t) \subsetneq \sigma(\mathfrak{a}^{t-\varepsilon})$ for all $\varepsilon > 0$. We claim that $t = \operatorname{ct}_{\mathcal{C}}^{\sigma(M,\mathcal{C},\mathfrak{a}^t)}(\mathfrak{a})$. Let $e \gg 0$ be such that $\sigma(M,\mathcal{C},\mathfrak{a}^t) = \mathcal{C}^e(\mathfrak{a}^{\lceil tp^e \rceil} M)$. Then $\mathcal{C}^e(\mathfrak{a}^{\lceil tp^e \rceil} M) \subseteq \sigma(M,\mathcal{C},\mathfrak{a}^t)$, and so, $\delta_{\mathfrak{a}}^{N,\mathcal{C}}(p^e) < \lceil tp^e \rceil$. Thus, $\operatorname{ct}_{\mathcal{C}}^{\sigma(M,\mathcal{C},\mathfrak{a}^t)}(\mathfrak{a}) \leqslant t$. By way of contradiction, assume that the inequality were strict. In this case, $\operatorname{ct}_{\mathcal{C}}^{\sigma(M,\mathcal{C},\mathfrak{a}^t)}(\mathfrak{a}) + \varepsilon \leqslant t - \varepsilon$ for some $\varepsilon > 0$. If we set $\alpha = \operatorname{ct}_{\mathcal{C}}^{\sigma(M,\mathcal{C},\mathfrak{a}^t)}(\mathfrak{a})$, we then have a containment $\mathfrak{a}^{\lceil (\alpha+\varepsilon)p^e \rceil} \supseteq \mathfrak{a}^{\lceil (t-\varepsilon)p^e \rceil}$, and

$$\mathcal{C}^e(\mathfrak{a}^{\lceil (\alpha+\varepsilon)p^e \rceil} M) \supseteq \mathcal{C}^e(\mathfrak{a}^{\lceil (t-\varepsilon)p^e \rceil} M) = \sigma(M,\mathcal{C},\mathfrak{a}^{t-\varepsilon}) \supsetneq \sigma(M,\mathcal{C},\mathfrak{a}^t).$$

for $e \gg 0$. In particular, there exists $\varphi \in \mathcal{C}^e$, an element $f \in \mathfrak{a}^{\lceil(\alpha+\varepsilon)p^e\rceil}$, and $m \in M$ such that $\varphi(fm) \notin \sigma(M, \mathcal{C}, \mathfrak{a}^t)$. Since for $e \gg 0$, we have $(\alpha+\varepsilon)p^e \geqslant \alpha p^e + 1 \geqslant \delta_{\mathfrak{a}}^{\sigma(M,\mathcal{C},\mathfrak{a}^t),\mathcal{C}}(p^e)+1$, we see that $\mathfrak{a}^{\delta_{\mathfrak{a}}^{\sigma(M,\mathcal{C},\mathfrak{a}^t),\mathcal{C}}(p^e)+1} \not\subseteq I_e(\sigma(M,\mathcal{C},\mathfrak{a}^t),\mathcal{C})$, a contradiction. Therefore the jumping number t equals the threshold $\operatorname{ct}_{\mathcal{C}}^{\sigma(M,\mathcal{C},\mathfrak{a}^t)}(\mathfrak{a})$. □

4. Principally Generated Cartier Algebras

Throughout this section, $(R, \mathfrak{m}, K)$ is a local ring, M is a finitely generated R-module, and $\mathcal{C}$ is a Cartier algebra on M.

4.1. **Rationality for σ-jumping numbers.**

Example 4.1. Suppose that R is an F-finite and F-injective Cohen-Macaulay local ring. If $M = \omega_R$ is a canonical module for R, then the full Cartier algebra on ω_R is principally generated by the trace map $\Phi \in \mathcal{C}^1_{\omega_R}$ [LS01, Proposition 3.10 (1)]. Furthermore, under these assumptions Φ is surjective because its Matlis dual is the Frobenius action on $H^d_{\mathfrak{m}}(R)$. In particular, this applies to R if the ring itself when R is Gorenstein.

We proceed to prove statements that are inspired by the behavior of test ideals in Gorenstein rings.

Proposition 4.2. *Let $\mathfrak{a}$ be an ideal of R, M be a finitely generated R-module and $\mathcal{C} = R[\Phi]$ be a principal Cartier subalgebra of $\mathcal{C}_M$. If t is a jumping number of $\sigma(M, \mathcal{C}, \mathfrak{a}^\bullet)$, then so is $p \cdot t$.*

Proof. Fix R-modules $N \subseteq M$ for which $t = \operatorname{ct}_{\mathcal{C}}^N(\mathfrak{a})$, and for all $e \geqslant 0$, let $I_e = I_e(N)$. We claim that $p \cdot t = \operatorname{ct}_{\mathcal{C}}^{I_1}(\mathfrak{a})$. If $J_e = \operatorname{ann}_R(M/I_e)$, we have that $t = \lim_{e\to\infty} \frac{c^{J_e}(\mathfrak{a})}{p^e}$ by Theorem 3.4. Note that

$$p \cdot t = p \cdot \lim_{e\to\infty} \frac{c^{J_e}(\mathfrak{a})}{p^e} = \lim_{e\to\infty} \frac{c^{J_e}(\mathfrak{a})}{p^e}.$$

We claim that $I_{e+1} = I_e(I_1)$. In fact, the containment $I_{e+1} \subseteq I_e(I_1)$ is always true. For the converse, fix $x \in I_e(I_1)$. Since Φ is a generator of $\mathcal{C}$, it is enough to show that $\Phi^{e+1}(xM) \subseteq N$. We have that

$$\Phi^{e+1}(xM) = \Phi\left(\Phi^e(xM)\right) \subseteq N$$

because $\Phi^e(xM) \subseteq I_1$ by assumption. Hence, again applying Theorem 3.4,

$$p \cdot t = \lim_{e\to\infty} \frac{c^{J_{e+1}}(\mathfrak{a})}{p^e} = \lim_{e\to\infty} \frac{c^{\operatorname{ann}_R(M/I_e(I_1))}(\mathfrak{a})}{p^e} = \operatorname{ct}_{\mathcal{C}}^{I_1}(\mathfrak{a}).$$ □

We are now prepared to prove one of our main results, showing that under relatively mild assumptions, the σ-jumping numbers are rational numbers. Our proof does not apply (or justify) the fact that the set of σ-jumping numbers for an ideal form a discrete set.

Theorem 4.3. *Let $\mathfrak{a}$ be an ideal of R, M a finitely generated R-module, and $\mathcal{C} = R[\Phi]$ be a principal Cartier subalgebra of $\mathcal{C}_M$. If* $\operatorname{cpt}(M, \mathcal{C}, \mathfrak{a}) > 0$*, then every jumping number of $\sigma(M, \mathcal{C}, \mathfrak{a}^\bullet)$ is rational.*

Proof. By Proposition 2.20, it suffices to prove the statement for jumping numbers $t \in [0, \mu(\mathfrak{a})]$. If for some $e \geqslant 0$, $p^e t$ is an integer, then $t \in \mathbb{Q}$.

Otherwise, $p^e t \notin \mathbb{Z}$ for all $e \geqslant 0$. If $p^e t \geqslant \mu(\mathfrak{a})$, then $p^e t - \lfloor p^e t \rfloor + \mu(\mathfrak{a}) - 1$ is also a jumping number by Propositions 2.20 and 4.2. Fix a sequences of nonnegative integers $\{e_i\}_{i=0}^{\infty}$ and real numbers $\{t_i\}_{i=0}^{\infty}$ inductively as follows: We set $e_0 = 0$ and $t_0 = t$. Given e_i and t_i, let e_{i+1} be an integer greater than e_i such that $p^{e_{i+1}} t \geqslant \mu(\mathfrak{a})$, and let $t_{i+1} = p^e t - \lfloor p^e t \rfloor + \mu(\mathfrak{a}) - 1 \in [0, \mu(\mathfrak{a})]$. Since $\{t_i \mid i \geqslant 0\} \subseteq [0, 1]$ is a finite by Theorem 2.22, there exists $j \neq s$ such that $t_j = t_s$. Hence

$$p^e t - \lfloor p^e t \rfloor + \mu(\mathfrak{a}) - 1 = p^s t - \lfloor p^s t \rfloor + \mu(\mathfrak{a}) - 1,$$

so that $t = \frac{\lfloor p^e t \rfloor - \lfloor p^s t \rfloor}{p^e - p^s} \in \mathbb{Q}$. □

We apply Theorem 4.3 in subsequent sections to show that several invariants of interest, whose rationality had not previously been established, are always rational.

4.2. **$\mathfrak{m}$-adic constancy for principal ideals.** The results contained in this subsection are motivated by the problem of $\mathfrak{m}$-adic constancy for F-pure thresholds. Specifically, for every $f \in R$, the questions is whether there exists an integer $N \geqslant 0$ such that

$$\operatorname{fpt}(f) = \operatorname{fpt}(f + h)$$

for every $h \in \mathfrak{m}^N$. This property is related to the ACC condition for F-pure thresholds, which was recently established for strongly F-regular rings [Sat19] and for ideals with fixed embedding dimension [Sat21]. In what follows, we prove $\mathfrak{m}$-adic constancy for the first σ-jumping number. This proof generally follows the same line as work on F-pure thresholds of isolated singularities [HNBW18]. However, for this result, we employ homological methods and base p expansions in order to work in a more general setting.

We first recall some notation and basic facts on base p expansions. Given $t \in (0, 1]$, there exist unique integers $t^{(e)}$ for every $e \geqslant 1$ such that $0 \leqslant t^{(e)} \leqslant p - 1$, and $t = \sum_{e \geqslant 1} t^{(e)} p^{-e}$, and such that the integers $t^{(e)}$ are not eventually zero. The integer $t^{(e)}$ is called the e-th digit of t (base p), and $t = \sum_{e \geqslant 1} t^{(e)} p^{-e}$ is called the non-terminating base p expansion of t.

Definition 4.4. Fix $t \in (0, 1]$ and an integer $e \geqslant 1$. We define the *e-th truncation* of t (base p) as $\langle t \rangle_e = t^{(1)} p^{-1} + \ldots + t^{(e)} p^{-e}$, and sometimes write $\langle t \rangle_e = .\ t^{(1)} : t^{(2)} : \ldots : t^{(e)}$ (base p).

Now suppose that $(R, \mathfrak{m}, K)$ is a complete local ring, M a finitely generated R-module with a principal Cartier subalgebra $\mathcal{C} = R[\Phi]$ of $\mathcal{C}_M$. Observe that $M^\vee$, the Matlis dual of M, is an Artinian R-module with an injective Frobenius action F induced by Φ. Let

$f \in R$. For every integer $e > 0$ and all $\nu_1, \ldots, \nu_e \in \mathbb{N}$ such that $0 \leqslant \nu_i \leqslant p-1$, the composition of the maps

$$M^\vee \xrightarrow{f^{\nu_1} F} M^\vee \xrightarrow{f^{\nu_2} F} M^\vee \longrightarrow \cdots \xrightarrow{f^{\nu_{e-1}} F} M^\vee \xrightarrow{f^{\nu_e} F} M^\vee$$

equals the map $f^u F^e$, where $u = p^{e-1}\nu_1 + p^{e-2}\nu_2 + \ldots + p\nu_{e-1} + \nu_e$. In particular, we have

$$\ker(f^{p\langle t\rangle_1} F) \subseteq \ker(f^{p^2\langle p\rangle_2} F^2) \subseteq \ldots \subseteq \ker(f^{p^e\langle t\rangle_e} F^e) \subseteq \ker(f^{p^{e+1}\langle p\rangle_{e+1}} F^{e+1}) \subseteq \ldots$$

for every $t \in (0,1]$.

Lemma 4.5. *Let $(R, \mathfrak{m}, K)$ be a complete local ring, and M a finitely generated R-module with a surjective Cartier map Φ. Let F be the induced Frobenius map on $M^\vee$. Let $f \in R$, and $t \in (0,1]$. Then*

$$\left(M^\vee / \ker(f^{p^e\langle t\rangle_e} F^e)\right)^\vee \cong \Phi^e(f^{p^e\langle t\rangle_e} M),$$

Proof. Applying Matlis duality to the exact sequence

$$0 \to \ker(f^{p^e\langle t\rangle_e} F^e) \to M^\vee \xrightarrow{f^{p^e\langle t\rangle_e} F^e} M^\vee$$

we obtain

$$M \xrightarrow{\Phi^e(f^{p^e\langle t\rangle_e} -)} M \to \left(\ker(f^{p^e\langle t\rangle_e} F^e)\right)^\vee \to 0.$$

Therefore, we see that $\left(\ker(f^{p^e\langle t\rangle_e} F^e)\right)^\vee \cong M/\Phi^e(f^{p^e\langle t\rangle_e} M)$. Furthermore, applying Matlis duality to the short exact sequence

$$0 \to \ker(f^{p^e\langle t\rangle_e} F^e) \to M^\vee \to M^\vee / \ker(f^{p^e\langle t\rangle_e} F^e) \to 0,$$

we obtain

$$0 \to \left(M^\vee / \ker(f^{p^e\langle t\rangle_e} F^e)\right)^\vee \to M \to M/\Phi^e(f^{p^e\langle t\rangle_e} M) \to 0.$$

It follows that $\left(M^\vee / \ker(f^{p^e\langle t\rangle_e} F^e)\right)^\vee \cong \Phi^e(f^{p^e\langle t\rangle_e} M)$, as desired. $\square$

An immediate consequence is the following corollary.

Corollary 4.6. *Let $(R, \mathfrak{m}, K)$ be a local ring, and M a finitely generated R-module with a surjective Cartier map Φ. Let $f \in R$, and $t > 0$. Then for all $e, e' > 0$ we have $\ker(f^{p^e\langle t\rangle_e} F^e) = \ker(f^{p^{e'}\langle t\rangle_{e'}} F^{e'})$ if and only if $\Phi^e(f^{p^e\langle t\rangle_e} M) = \Phi^{e'}(f^{p^{e'}\langle t\rangle_{e'}} M)$.*

Proof. Since both equalities hold if and only if they are true under completion, we may assume that R is complete. Suppose that $e' > e$, then Lemma 4.5 gives

$$\left(\operatorname{coker}\left(\ker(f^{p^{e'}\langle t\rangle_{e'}} F^{e'}) \hookrightarrow \ker(f^{p^e\langle t\rangle_e} F^e)\right)\right)^\vee = \ker\left(\frac{M}{\Phi^e(f^{p^e\langle t\rangle_e} M)} \to \frac{M}{\Phi^{e'}(f^{p^{e'}\langle t\rangle_{e'}} M)}\right),$$

and the claim now follows. $\square$

We now recall the Hartshorne-Speiser-Lyubeznik Theorem [HS77, Lyu97] for Frobenius actions on Artinian modules (see also [Sha07] for a short proof).

Theorem 4.7 ([HS77, Lyu97])**.** *Let $(R, \mathfrak{m}, K)$ be a local ring. Let H be an Artinian R-module that has a Frobenius action F. Then there exists $e > 0$ such that $F^e(\Gamma_F(H)) = 0$, where $\Gamma_F(H) = \{x \in H \mid F^{e'}(x) = 0$ for some $e' \in \mathbb{N}\}$.*

The Hartshorne-Speiser-Lyubeznik Theorem implies the following statement: for $r, e \in \mathbb{N}$ with $r < p^e =: q$, there exists $B \in \mathbb{N}$ such that

$$\ker(f^{q^{B-1}r+\ldots+r}F^{eB}) = \ker((f^rF^e)^B) = \ker((f^rF^e)^{B+\ell}) = \ker(f^{q^{B+\ell-1}r+\ldots+r}F^{e(B+\ell)})$$

for all $\ell \geqslant 0$. Therefore, for all ℓ, and $r < q$ we obtain that

$$\Phi^{eB}(f^{q^{N-1}r+\ldots+qr+r}M) = \Phi^{e(B+\ell)}(f^{q^{B+\ell-1}r+\ldots+qr+r}M)$$

The following lemma summarizes this fact, and extends it to the non-local case.

Lemma 4.8. *Let $(R, \mathfrak{m}, K)$ be a local ring, and M be a finitely generated R-module with a surjective Cartier map Φ. Let F be the induced Frobenius map on $M^\vee$. Let $f \in R$, and let $r, e \in \mathbb{N}$ be such that $r < q := p^e$. Then, there exists $B \in \mathbb{N}$ such that for all $\ell \geqslant 0$*

$$\Phi^{eB}(f^{q^{B-1}r+\ldots+qr+r}M) = \Phi^{e(B+\ell)}(f^{q^{B+\ell-1}r+\ldots+qr+r}M)$$

Proof. For each $\mathfrak{p} \in \operatorname{Spec}(R)$ there exists $B_\mathfrak{p}$ for which

$$\Phi^{eB}(f^{q^{B-1}r+\ldots+qr+r}M)_\mathfrak{p} = \Phi^{e(B+\ell)}(f^{q^{B+\ell-1}r+\ldots+qr+r}M)_\mathfrak{p}$$

for all $\ell \geqslant 0$. Therefore, for each $\mathfrak{p}$ there exists $h_\mathfrak{p} \in R \smallsetminus \mathfrak{p}$ such that

$$\Phi^{eB}(f^{q^{B-1}r+\ldots+qr+r}M)_{h_\mathfrak{p}} = \Phi^{e(B+\ell)}(f^{q^{B+\ell-1}r+\ldots+qr+r}M)_{h_\mathfrak{p}}.$$

Since $\{D(h_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\}$ forms an open cover of the compact space $\operatorname{Spec}(R)$, we can find a finite sub-cover $\{D(h_1), \ldots, D(h_t)\}$, where each $h_i \in R \smallsetminus \mathfrak{p}_i$ for some $\mathfrak{p}_i \in \operatorname{Spec}(R)$. By letting $B = \max\{B_{\mathfrak{p}_1}, \ldots, B_{\mathfrak{p}_t}\}$, we have that

$$\Phi^{eB}(f^{q^{B-1}r+\ldots+qr+r}M)_\mathfrak{p} = \Phi^{e(B+\ell)}(f^{q^{B+\ell-1}r+\ldots+qr+r}M)_\mathfrak{p}.$$

for all $\mathfrak{p} \in \operatorname{Spec}(R)$. Then, the equality holds for these modules before localizing at $\mathfrak{p}$. ☐

We now introduce some notation. For $t > 0$, we let $\sigma(M, \mathcal{C}, f^{t-\varepsilon}) := \bigcap_{s<t} \sigma(M, \mathcal{C}, f^s)$. Note that $\sigma(M, \mathcal{C}, f^{t-\varepsilon}) \supseteq \sigma(M, \mathcal{C}, f^t)$ always holds true, and the containment is strict if and only if t is a jumping number of f.

Proposition 4.9. *Let M be a finitely generated R-module and $\mathcal{C} = R[\Phi]$ be a principal Cartier subalgebra of $\mathcal{C}_M$. Let $t \in (0, 1]$ be a rational number, whose non-terminating base p expansion is $0.s_1 : \ldots s_a : \overline{r_1 : \ldots : r_b}$. Let $q = p^b$, and $\gamma = 0.\overline{r_1 : \ldots : r_b}$. If*

$$\Phi^{bN}(f^{q^N\langle\gamma\rangle_{bN}}M) = \Phi^{bN+1}(f^{q^{(N+1)}\langle\gamma\rangle_{b(N+1)}}M)$$

for some $N \in \mathbb{N}$, then

$$\sigma(M, \mathcal{C}, f^{t-\varepsilon}) = \Phi^{a+bN}(f^{p^a q^N\langle t\rangle_{a+bN}}M).$$

Proof. Since σ-ideals commute with completion, we may assume that R is complete. Let $s = p^a\langle t\rangle_a$, and $r = p^b\langle\gamma\rangle_b$. Consider the maps $\varphi = f^sF^a$ and $\phi = f^rF^b$, which are endomorphisms of $M^\vee$. Because of Corollary 4.6, our assumptions imply that $\ker(\phi^N) = \ker(\phi^{N+1})$. Then, for every $j \geqslant 0$ we have

$$\ker(\phi^{N+j+1}) = (\phi^j)^{-1}(\ker(\phi^{N+1})) = (\phi^j)^{-1}(\ker(\phi^N)) = \ker(\phi^{N+j}),$$

and thus $\ker(\phi^N) = \ker(\phi^{N+j})$. As a consequence, for all $j \geqslant 0$ we have

$$\ker(\phi^{N+j}\varphi) = \varphi^{-1}\ker(\phi^{N+j}) = \varphi^{-1}\ker(\phi^N) = \ker(\phi^N\varphi),$$

and again by Corollary 4.6 we conclude that

$$\Phi^{a+bN}(f^{p^{a+bN}\langle t\rangle_{a+bN}}M) = \Phi^{a+b(N+j)}(f^{p^{a+b(N+j)}\langle t\rangle_{a+b(N+j)}}M)$$

for all $j \geqslant 0$. Therefore, since for $e \in \mathbb{N}$ the truncations $\langle t\rangle_{a+be}$ are strictly less that t and they converge to t, we finally obtain that

$$\begin{aligned}\sigma(M,\mathcal{C},f^{t-\varepsilon}) &= \bigcap_{\lambda<t}\sigma(M,\mathcal{C},f^\lambda) \text{ by definition}\\ &= \bigcap_{e\in\mathbb{N}}\sigma(M,\mathcal{C},f^{\langle t\rangle_{a+be}}) \text{ because } \langle t\rangle_{a+be}\to t \text{ from below}\\ &= \bigcap_{e\in\mathbb{N}}\Phi^{a+be}(f^{p^{a+be}\langle t\rangle_{a+be}}M) \text{ by Proposition 2.7(7)}\\ &= \Phi^{a+bN}(f^{p^{a+bN}\langle t\rangle_{a+bN}}M).\end{aligned}$$

□

Lemma 4.10. *Let $(R,\mathfrak{m},K)$ be a complete local ring, M a finitely generated R-module, and $\mathcal{C} = R[\Phi]$ be a principal Cartier subalgebra of $\mathcal{C}_M$. For a fixed $f \in R$, suppose that $M/\sigma(M,\mathcal{C},f^{1-\varepsilon})$ has finite length ℓ as an R-module. Then for every σ-jumping number $t \in (0,1)$, $p^a(p^b-1)t \in \mathbb{N}$ for some integers $a,b \geqslant 0$ such that $a+b \leqslant \ell$.*

Proof. For $1 \leqslant j \leqslant \ell$, let $t_j = p^jt - \lfloor p^jt\rfloor$. If $t_j = 0$ for some j, then $p^jt \in \mathbb{N}$ and the result follows. Otherwise, each t_j is a positive σ-jumping number by Propositions 4.2 and 2.20. Since there are at most ℓ jumping numbers in $(0,1)$, $t_i = t_j$ for some $0 \leqslant i < j \leqslant \ell$. It follows that $p^jt - p^it = p^{j-i}(p^i-1)t = \lfloor p^jt\rfloor - \lfloor p^it\rfloor \in \mathbb{N}$, as desired. □

Theorem 4.11. *Let $(R,\mathfrak{m},K)$ be a complete local ring, M a finitely generated R-module, and $\mathcal{C} = R[\Phi]$ be a principal Cartier subalgebra of $\mathcal{C}_M$. For a fixed $f \in R$, suppose that $M/\sigma(M,\mathcal{C},f^{1-\varepsilon})$ has finite length as an R-module. Then there exists an integer $N \geqslant 0$ such that*

$$\sigma(M,\mathcal{C},f^t) = \sigma(M,\mathcal{C},(f+h)^t)$$

for all $h \in \mathfrak{m}^{[p^N]}$ and all $t \in (0,1)$. In particular, the sets of σ-jumping numbers of $\sigma(M,\mathcal{C},f^\bullet)$ and $\sigma(M,\mathcal{C},(f+h)^\bullet)$ coincide.

Proof. Let ℓ be the length of $M/\sigma(M,\mathcal{C},f^{1-\varepsilon})$, and set $N = \ell^2+3\ell+2$. Then $\mathfrak{m}^{[p^\ell]}M \subseteq \sigma(M,\mathcal{C},f^{1-\varepsilon})$, and $\sigma(M,\mathcal{C},f^{1-\varepsilon}) = \Phi^\ell(f^{p^\ell-1}M)$ by Proposition 4.9. Then for $h \in \mathfrak{m}^{[p^N]}$,

$$\begin{aligned}\Phi^{\ell+1}((f+h)^{p^{\ell+1}-1}M) &\subseteq \Phi^\ell((f+h)^{p^\ell-1}M) \subseteq \Phi^\ell(f^{p^\ell-1}M) + \Phi^\ell(hM)\\ &\subseteq \sigma(M,\mathcal{C},f^{1-\varepsilon}) + \mathfrak{m}^{[p^{N-\ell}]}M \subseteq \sigma(M,\mathcal{C},f^{1-\varepsilon}).\end{aligned}$$

Fix generators $w_1, \ldots, w_s$ for $\sigma(M, \mathcal{C}, f^{1-\varepsilon})$, and for $1 \leqslant i \leqslant s$, choose $v_i \in M$ for which $\Phi^{\ell+1}(f^{p^{\ell+1}-1} v_i) = w_i$. Then

$$\Phi^{\ell+1}((f+h)^{p^{\ell+1}-1} v_i) \in \Phi^{\ell+1}(f^{p^{\ell+1}-1} v_i) + \Phi^{\ell+1}(hM) = w_i + \Phi^{\ell+1}(hM).$$

Observe that $\Phi^{\ell+1}(hM) \subseteq \mathfrak{m}^{[p^{N-\ell-1}]} M \subseteq \mathfrak{m}^{\ell+1} M \subseteq \mathfrak{m}\sigma(M, \mathcal{C}, f^{1-\varepsilon})$, and by Nakayama's Lemma, $\{\Phi^{\ell+1}((f+h)^{p^{\ell+1}-1} v_i) \mid 1 \leqslant i \leqslant s\}$ generates $\sigma(M, \mathcal{C}, f^{1-\varepsilon})$. Thus, we have that

$$\sigma(M, \mathcal{C}, f^{1-\varepsilon}) \subseteq \Phi^{\ell+1}((f+h)^{p^{\ell+1}-1} M) \subseteq \Phi^{\ell}((f+h)^{p^{\ell}-1} M) \subseteq \sigma(M, \mathcal{C}, f^{1-\varepsilon}).$$

In particular, we see that $\Phi^{\ell+1}((f+h)^{p^{\ell+1}-1} M) = \Phi^{\ell}((f+h)^{p^{\ell}-1} M)$, and it follows by Proposition 4.9 that $\sigma(M, \mathcal{C}, (f+h)^{1-\varepsilon}) = \Phi^{\ell}((f+h)^{p^{\ell}-1} M)$. We conclude that $\sigma(M, \mathcal{C}, f^{1-\varepsilon}) = \sigma(M, \mathcal{C}, (f+h)^{1-\varepsilon})$.

Now fix $t \in (0,1)$. If t is a σ-jumping number of f, or of $f+h$, then by Lemma 4.10, $t = \frac{r}{p^a(p^b-1)}$ for some integers $r, a, b \geqslant 0$ such that $a + b \leqslant \ell$. Since ℓ is also an upper bound for the length of $M/\sigma(M, \mathcal{C}, f^{t-\varepsilon})$, we have that $\Phi^{b(\ell+1)}(f^{(p^{b(\ell+1)-1}+\ldots+1)r} M) = \Phi^{b\ell}(f^{(p^{b\ell-1}+p^{b\ell}+\ldots+1)r} M)$, and thus $\sigma(M, \mathcal{C}, f^{t-\varepsilon}) = \Phi^{a+bn}(f^{p^a(p^{bn-1}+\ldots+1)r} M)$ for all $n \geqslant \ell$ by Proposition 4.9. Observe that

$$\begin{aligned}
\Phi^{a+b\ell}((f+h)^{(p^{a+b\ell-1}+\ldots+1)r} M) &\subseteq \Phi^{a+b\ell}(f^{(p^{a+b\ell-1}+\ldots+1)r} M) + \Phi^{a+b\ell}(hM) \\
&\subseteq \Phi^{a+b\ell}(f^{p^a(p^{b\ell-1}+\ldots+1)r} M) + \mathfrak{m}^{[p^{N-a-b\ell}]} M \\
&\subseteq \sigma(M, \mathcal{C}, f^{t-\varepsilon}) + \mathfrak{m}^{[p^{\ell}]} M \\
&\subseteq \sigma(M, \mathcal{C}, f^{t-\varepsilon}) + \sigma(M, \mathcal{C}, f^{1-\varepsilon}) \\
&\subseteq \sigma(M, \mathcal{C}, f^{t-\varepsilon}).
\end{aligned}$$

Let $\alpha = \lceil \frac{a}{b} \rceil$, and similar to above, fix generators $w_1, \ldots, w_s$ for$\sigma(M, \mathcal{C}, f^{t-\varepsilon})$, and for $1 \leqslant i \leqslant s$, choose $v_i \in M$ for which $\Phi^{b(\ell+\alpha+1)}(f^{(p^{b(\ell+\alpha+1)-1}+\ldots+1)r} v_i) = w_i$. Then

$$\begin{aligned}
\Phi^{b(\ell+\alpha+1)}((f+h)^{(p^{b(\ell+\alpha+1)-1}+\ldots+1)r} v_i) &\in \Phi^{b(\ell+\alpha+1)}(f^{(p^{b(\ell+\alpha+1)-1}+\ldots+1)r} v_i) + \Phi^{b(\ell+\alpha+1)}(hM) \\
&\subseteq w_i + \mathfrak{m}^{[p^{N-b(\ell+\alpha+1)}]} M \subseteq w_i + \mathfrak{m}\sigma(M, \mathcal{C}, f^{t-\varepsilon})
\end{aligned}$$

where the last containment follows from the fact that $\ell^2 + 3\ell + 2 - b(\ell+\alpha+1) \geqslant \ell + 1$, and thus $\Phi^{b(\ell+\alpha+1)}(hM) \subseteq \mathfrak{m}^{\ell+1} M \subseteq \mathfrak{m}\sigma(M, \mathcal{C}, f^{1-\varepsilon}) \subseteq \mathfrak{m}\sigma(M, \mathcal{C}, f^{t-\varepsilon})$. It follows from Nakayama's Lemma that $\{\Phi^{b(\ell+\alpha+1)}((f+h)^{(p^{b(\ell+\alpha+1)-1}+\ldots+1)r} v_i) \mid 1 \leqslant i \leqslant s\}$ is a set of generators for $\sigma(M, \mathcal{C}, f^{t-\varepsilon})$. Then since $b(\ell+\alpha) \geqslant a + b\ell$, we have that

$$\begin{aligned}
\sigma(M, \mathcal{C}, f^{t-\varepsilon}) &\subseteq \Phi^{b(\ell+\alpha+1)}((f+h)^{(p^{b(\ell+\alpha+1)-1}+\ldots+1)r} M) \\
&\subseteq \Phi^{b(\ell+\alpha)}((f+h)^{(p^{b(\ell+\alpha)-1}+\ldots+1)r} M) \\
&\subseteq \Phi^{a+b\ell}((f+h)^{(p^{a+b\ell-1}+\ldots+1)r} M) \\
&\subseteq \sigma(M, \mathcal{C}, f^{t-\varepsilon}),
\end{aligned}$$

forcing all inclusions to be equalities. By Proposition 4.9 we conclude that for $n \geqslant \ell + \alpha$, $\sigma(M, \mathcal{C}, (f+h)^{t-\varepsilon}) = \Phi^{a+bn}((f+h)^{p^a(p^{bn-1}+\ldots+1)r} M)$. Taking $n = \ell + \alpha$ and applying Nakayama's Lemma once more,

$$\sigma(M, \mathcal{C}, f^{t-\varepsilon}) \subseteq \Phi^{a+bn}((f+h)^{(p^{a+bn-1}+\ldots+1)r} M) \subseteq \Phi^{a+bn}((f+h)^{p^a(p^{bn-1}+\ldots+1)r} M).$$

Therefore,

$$\begin{aligned}\sigma(M,\mathcal{C},f^{t-\varepsilon}) &\subseteq \Phi^{a+bn}((f+h)^{p^a(p^{bn-1}+\ldots+1)r}M)\\ &\subseteq \Phi^{a+bn}(f^{p^a(p^{bn-1}+\ldots+1)r}M)+\Phi^{a+bn}(hM)\\ &\subseteq \sigma(M,\mathcal{C},f^{t-\varepsilon})+\mathfrak{m}^{[p^{\ell+1}]}M\subseteq \sigma(M,\mathcal{C},f^{t-\varepsilon}).\end{aligned}$$

Since $\Phi^{a+bn}((f+h)^{p^a(p^{bn-1}+\ldots+1)r}M)=\sigma(M,\mathcal{C},(f+h)^{t-\varepsilon})$, the proof is complete. □

5. Applications to Useful Numerical Invariants

In this section, we discuss relations between F-pure submodules and Cartier thresholds with other types of test modules and their numerical invariants. For example, we compare F-pure submodules with certain test ideals and test modules [Sch11, Bli13], and we relate Cartier thresholds to numerical invariants defined in terms of local cohomology [STV17], and to F-thresholds [MTW05, HMTW08].

5.1. **Relations with test modules and F-jumping numbers.** Here, we compare F-pure submodules with test ideals.

Proposition 5.1. *Fix an ideal $\mathfrak{a}$ of a Noetherian ring R of prime characteristic $p>0$, and a real number $t>0$. Let M be a finitely generated R-module, and $\mathcal{C}$ a Cartier algebra on M. Assume that $\tau(M,\mathcal{C},\mathfrak{a}^t)$ exists, and that $\bigcup_{e\geqslant 0}\mathcal{C}^e(\tau(M,\mathcal{C},\mathfrak{a}^t))=M$. Then $\sigma(M,\mathcal{C}_M,\mathfrak{a}^t)=\tau(M,\mathcal{C},\mathfrak{a}^t)$.*

Proof. Choose $e'\gg 0$ so that $\mathcal{C}_M^{e'}(\tau_b(\mathfrak{a}^t,\mathcal{C}))=M$. For $e\gg 0$, we then have

$$\begin{aligned}\sigma(M,\mathcal{C},\mathfrak{a}^t) &= \mathcal{C}_M^e(\mathfrak{a}^{\lceil tp^e\rceil}M)\\ &\subseteq \mathcal{C}^e(\mathfrak{a}^{\lceil tp^e\rceil}\mathcal{C}_M^{e'}(\tau_b(\mathfrak{a}^t,\mathcal{C})))\\ &\subseteq \mathcal{C}_M^{e+e'}(\mathfrak{a}^{\lceil tp^{e+e'}\rceil}\tau(M,\mathcal{C}_M,\mathfrak{a}^t))\\ &\subseteq \tau(M,\mathcal{C}_M,\mathfrak{a}^t).\end{aligned}$$

As noted by Blickle [Bli13, Definition 3.1], the reverse inclusion always holds. □

5.2. **Numerical invariants from local cohomology.** In this subsection, we prove that certain numerical invariants introduced by Singh, Takagi, and Varbaro to study F-injectivity on local cohomology are rational numbers [STV17]. In the graded case, these numbers are closely related to the a-invariant.

Definition 5.2. Let $(R,\mathfrak{m},K)$ be a local ring of prime characteristic $p>0$, and $\mathfrak{a}$ an ideal of R such that $\mathfrak{a}\cap R^\circ\neq\varnothing$. Given a real number $t\geqslant 0$ and an integer $i\geqslant 0$, we define the $\mathfrak{a}^t$*-sharp Frobenius closure* of 0 in $H^i_\mathfrak{m}(R)$ as

$$0^{F\#\mathfrak{a}^t}_{H^i_\mathfrak{m}(R)}=\{v\in H^i_\mathfrak{m}(R)\mid \mathfrak{a}^{\lceil t(p^e-1)\rceil}F^e_{H^i_\mathfrak{m}(R)}(v)=0 \text{ for } e\gg 0\}.$$

and the $\mathfrak{a}^t$*-Frobenius closure* of 0 in $H^i_\mathfrak{m}(R)$ is

$$0^{F\mathfrak{a}^t}_{H^i_\mathfrak{m}(R)}=\{v\in H^i_\mathfrak{m}(R)\mid \mathfrak{a}^{\lceil tp^e\rceil}F^e_{H^i_\mathfrak{m}(R)}(v)=0 \text{ for } e\gg 0\}.$$

Definition 5.3 ([STV17, ST08])**.** Let $(R, \mathfrak{m}, K)$ be an F-finite F-injective local ring of prime characteristic $p > 0$, and $\mathfrak{a}$ an ideal of R such that $\mathfrak{a} \cap R^\circ \neq \varnothing$. For each integer i, the *i-th F-injective threshold of* $\mathfrak{a}$, denoted by $\operatorname{fit}_i(\mathfrak{a})$, is defined by

$$\operatorname{fit}_i(\mathfrak{a}) = \sup\left\{t \in \mathbb{R}_{\geqslant 0} \mid 0^{F\#\mathfrak{a}^t}_{H^i_\mathfrak{m}(R)} = 0\right\}.$$

We first prove that these thresholds can be obtained from the Frobenius closure, instead of the sharp Frobenius closure.

Lemma 5.4. *Let $(R, \mathfrak{m}, K)$ be an F-finite F-injective local ring of prime characteristic $p > 0$, and $\mathfrak{a}$ an ideal of R such that $\mathfrak{a} \cap R^\circ \neq \varnothing$. Then*

$$\operatorname{fit}_i(\mathfrak{a}) = \sup\left\{t \in \mathbb{R}_{\geqslant 0} \mid 0^{F\mathfrak{a}^t}_{H^i_\mathfrak{m}(R)} = 0\right\}.$$

Proof. Let α denote the above supremum. For $v \in 0^{F\#\mathfrak{a}^t}_{H^i_\mathfrak{m}(R)}$, we have that $\mathfrak{a}^{\lceil t(p^e-1)\rceil} F^e_{H^i_\mathfrak{m}(R)}(v) = 0$, and so $\mathfrak{a}^{\lceil tp^e\rceil} F^e_{H^i_\mathfrak{m}(R)}(v) = 0$. Then $v \in 0^{F\mathfrak{a}^t}_{H^i_\mathfrak{m}(R)}$, so that $0^{F\mathfrak{a}^t}_{H^i_\mathfrak{m}(R)} = 0$, which implies that $0^{F\#\mathfrak{a}^t}_{H^i_\mathfrak{m}(R)} = 0$. Hence $\alpha \leqslant \operatorname{fit}_i(\mathfrak{a})$.

Now fix $t < \operatorname{fit}_i(\mathfrak{a})$, so that $0^{F\#\mathfrak{a}^t}_{H^i_\mathfrak{m}(R)} = 0$, meaning that the only $v \in H^i_\mathfrak{m}(R)$ satisfying $\mathfrak{a}^{\lceil t(p^e-1)\rceil} F^e_{H^i_\mathfrak{m}(R)}(v) = 0$ for $e \gg 0$ is $v = 0$. For $n > 0$, set $t_n = t(p^n - 1)/p^n$, and notice that for $e \geqslant n$, $t(p^e - 1) \geqslant t_n p^e$, so that $\lceil t(p^e - 1)\rceil \geqslant \lceil t_n p^e\rceil$. Now, assuming that $\mathfrak{a}^{\lceil t_n p^e)\rceil} F^e_{H^i_\mathfrak{m}(R)}(v) = 0$ for $e \gg 0$, we have that $\mathfrak{a}^{\lceil t(p^e-1)\rceil} F^e_{H^i_\mathfrak{m}(R)}(v) = 0$ for $e \gg 0$, forcing $v = 0$. Hence $0^{F\mathfrak{a}^{t_n}}_{H^i_\mathfrak{m}(R)} = 0$, so that $t_n \leqslant \alpha$ for every $n \geqslant 0$. Since $t_n \to t$, we have that $t \leqslant \alpha$. We conclude that $\operatorname{fit}_i(\mathfrak{a}) \leqslant \alpha$, and the two values must coincide. □

We now use Matlis duality to describe $\operatorname{fit}_i(\mathfrak{a})$ as a Cartier threshold.

Lemma 5.5. *Let $(R, \mathfrak{m}, K)$ be an F-finite F-injective complete local ring of prime characteristic $p > 0$, and let $\mathfrak{a}$ be an ideal of R. Let $\omega_i = (H^i_\mathfrak{m}(R))^\vee$, $\Phi_i = (F_{H^i_\mathfrak{m}(R)})^\vee \in \mathcal{C}^1_{\omega_i}$, and let $\mathcal{C}_i = R[\Phi_i]$. For every integer $e \geqslant 0$, the following facts are equivalent:*

1. $\Phi^e_i(\mathfrak{a}\omega_i) = \omega_i$.
2. $\ker(\mathfrak{a} F^e_{H^i_\mathfrak{m}(R)}) = 0$.

Proof. Fix generators $f_1, \ldots, f_\ell$ for $\mathfrak{a}$, and let θ denote the homomorphism $(\omega_i)^\ell \to \omega_i$ given by $(f_1, \ldots, f_\ell)$. Then composition $\Phi^e_i \circ \theta : (\omega_i)^\ell \to \omega_i$ is surjective if and only if the composition

$$H^i_\mathfrak{m}(R) \xrightarrow{F^e} H^i_\mathfrak{m}(R) \xrightarrow{\theta^T} (H^i_\mathfrak{m}(R))^\ell$$

is injective. The surjectivity of the first map is equivalent to $\Phi^e_i(\mathfrak{a}\omega_i) = \omega_i$, while the injectivity of the second map is equivalent to statement (2). □

We are now ready to state the main result of this subsection.

Theorem 5.6. *Let $(R, \mathfrak{m}, K)$ be an F-finite F-injective complete local ring of prime characteristic $p > 0$, and $\mathfrak{a} = (f)$ be a principal ideal of R such that $\mathfrak{a} \cap R^\circ \neq \varnothing$. Let*

$\omega_i = (H^i_{\mathfrak{m}}(R))^\vee$, $\Phi_i = (F_{H^i_{\mathfrak{m}}(R)})^\vee \in \mathcal{C}^1_{\omega_i}$, *and* $\mathcal{C}_i = R[\Phi_i]$. *Then* $\operatorname{fit}_i(\mathfrak{a})$ *is the first jumping number of* $(\omega_i, \mathcal{C}_i, \mathfrak{a})$.

Proof. Let $t \geqslant 0$ be a real number. Recall that $\sigma(\omega_i, \mathcal{C}_i, \mathfrak{a}^t) = \Phi_i^e(\mathfrak{a}^{\lceil tp^e \rceil}\omega_i)$ for all $e \gg 0$. Moreover, by Theorem 4.7 we have that $0^{F\mathfrak{a}^t}_{H^i_{\mathfrak{m}}(R)} = \ker(\mathfrak{a}^{\lceil tp^e \rceil} F^e)$ for all $e \gg 0$. By Lemma 5.5 we conclude that $\sigma(\omega_i, \mathcal{C}_i, \mathfrak{a}^t) = \omega_i$ if and only if $0^{F\mathfrak{a}^t}_{H^i_{\mathfrak{m}}(R)} = 0$, so that $\operatorname{fit}_i(\mathfrak{a})$ is the first jumping number of $(\omega_i, \mathcal{C}_i, \mathfrak{a})$ by Lemma 5.4. $\square$

Rationality of the F-injective thresholds is an immediate consequence of Theorem 5.6.

Corollary 5.7. *Let* $(R, \mathfrak{m}, K)$ *be an* F*-finite* F*-injective local ring of prime characteristic* $p > 0$*, and* $\mathfrak{a} = (f)$ *be a principal ideal of* R *such that* $\mathfrak{a} \cap R^\circ \neq \varnothing$. *If* $H^i_{\mathfrak{m}}(R) \neq 0$ *for some integer* $i \geqslant 0$*, then* $\operatorname{fit}_i(\mathfrak{a})$ *is a rational number.*

Proof. This follows from Theorems 4.3 and 5.6. $\square$

We turn to advancing the theory F-injectivity using the $\operatorname{fit}_i(\mathfrak{m})$. We use the notion of an F-full ring.

Definition 5.8 ([Bli01, Bli04, MQ18])**.** A local ring $(R, \mathfrak{m}, K)$ of prime characteristic $p > 0$ *F-full* if for every integer $i \geqslant 0$, the image of the Frobenius map on $H^i_{\mathfrak{m}}(R)$ generates $H^i_{\mathfrak{m}}(R)$ as an R-module.

Proposition 5.9 ([MQ18, Proposition 3.3 and 3.5])**.** *Let* $(R, \mathfrak{m}, K)$ *be a local ring of prime characteristic* $p > 0$*, and* $f \in \mathfrak{m}$ *a nonzerodivisor on* R. *If* R/fR *is* F*-full, then the map* $H^i_{\mathfrak{m}}(R) \xrightarrow{\cdot f} H^i_{\mathfrak{m}}(R)$ *is surjective for every integer* $i \geqslant 0$.

Proposition 5.10. *Let* $(R, \mathfrak{m}, K)$ *be an* F*-finite and* F*-injective local ring of prime characteristic* $p > 0$. *Let* $\omega_i = (H^i_{\mathfrak{m}}(R))^\vee$ *and* $\Phi_i = (F_{H^i_{\mathfrak{m}}(R)})^\vee \in \mathcal{C}^1_{\omega_i}$ *be the dual of the natural Frobenius action. Assume that* $\overline{R} = R/fR$ *is* F*-full for some* $f \in R$*, and let* $\overline{\omega_i} = (H^i_{\mathfrak{m}}(\overline{R}))^\vee$*, on which* $\overline{\Phi_i}$ *denotes the dual of the natural Frobenius action. Then the following hold:*

1. $\overline{\omega_{i-1}} = \omega_i / f\omega_i$.
2. *The* p^{-1}*-linear map* $\overline{\Phi}_{i-1}$ *is the one induced by the map* $\Phi_i(f^{p-1}-)$ *on* $\omega_i/f\omega_i$.
3. $\operatorname{fit}_i(f) = 1$ *if and only if* $\overline{\Phi}_{i-1}$ *is surjective.*

Proof. The short exact sequence, with Frobenius actions,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow{\ f\ } & R & \longrightarrow & \overline{R} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f^{p-1}F} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle \overline{F}} & & \\ 0 & \longrightarrow & R & \xrightarrow{\ f\ } & R & \longrightarrow & \overline{R} & \longrightarrow & 0 \end{array}$$

induces a long exact sequence in local cohomology, with Frobenius actions. By Proposition 5.9, the long exact sequence splits into exact sequences, for each i, of the following form.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{i-1}_{\mathfrak{m}}(\overline{R}) & \longrightarrow & H^i_{\mathfrak{m}}(R) & \xrightarrow{f} & H^i_{\mathfrak{m}}(R) & \longrightarrow & 0 \\ & & \downarrow \overline{F^e} & & \downarrow f^{p^e-1}F^e & & \downarrow F^e & & \\ 0 & \longrightarrow & H^{i-1}_{\mathfrak{m}}(\overline{R}) & \longrightarrow & H^i_{\mathfrak{m}}(R) & \xrightarrow{f} & H^i_{\mathfrak{m}}(R) & \longrightarrow & 0 \end{array}$$

Applying Matlis duality, we obtain that the (1). Since $\overline{F^e}$ is the restriction of $f^{p^e-1}F^e$ to $H^{i-1}_{\mathfrak{m}}(\overline{R}) \cong \operatorname{ann}_{H^i_{\mathfrak{m}}(R)}(f)$, (2) follows by Matlis duality.

For (3), observe that $\overline{\Phi_{i-1}}$ is surjective if and only if $\overline{F}$ is injective, which in turn holds if and only if $\overline{F^e}$ is injective for all integers $e > 0$. Since R is F-injective, we know that $\overline{F^e}$ is injective if and only if $f^{p^e-1}F^e$ is injective. Let $\mathcal{C}_i$ be the principal Cartier subalgebra of $\mathcal{C}_{\omega_i}$ generated by Φ_i. Then by Lemma 4.5, $\overline{\Phi_{i-1}}$ is surjective if and only if $\sigma(\omega_i, \mathcal{C}_i, f^{(p^e-1)/p^e}) = \Phi^e(f^{p^e-1}\omega_i) = \omega_i$ for all e, which is equivalent to the fact that $\sigma(\omega_i, \mathcal{C}_i, f^{1-\varepsilon}) = \omega_i$. Since $\sigma(\omega_i, \mathcal{C}_i, f^1) = f\omega_i$ by Proposition 2.7(7), 1 is the first σ-jumping number if and only if $\overline{R}$ is F-injective, and we are finished by Theorem 5.6. □

As an immediate consequence, we find that when R/fR is F-full, the support of $\omega_i/\sigma(\omega_i, \mathcal{C}_i, f^{1-\varepsilon})$ coincides with the non-F-injective locus of R/fR.

Lemma 5.11. *Let $(R, \mathfrak{m}, K)$ be an F-finite F-injective complete local ring of prime characteristic $p > 0$. Assume that R/fR is F-full for some $f \in R^\circ$. Given an integer $i \geqslant 0$, let $\omega_i = (H^i_{\mathfrak{m}}(R))^\vee$, $\Phi_i = (F_{H^i_{\mathfrak{m}}(R)})^\vee \in \operatorname{Hom}_1(R, R)$, and let $\mathcal{C}_i = R[\Phi_i] \subseteq \mathcal{C}_{\omega_i}$. Then for any prime ideal $\mathfrak{p}$ of R, $(\omega_i/\sigma(\omega_i, \mathcal{C}_i, f^{1-\varepsilon}))_{\mathfrak{p}} = 0$ if and only if $(R/fR)_{\mathfrak{p}}$ is F-injective.*

Proof. Suppose that $H^i_{\mathfrak{p}R_{\mathfrak{p}}}((R/fR)_{\mathfrak{p}}) \neq 0$ for some $i \geqslant 0$. Then

$$\begin{aligned} (\omega_{i+1}/\sigma(\omega_{i+1}, \mathcal{C}_{i+1}, f^{1-\varepsilon}))_{\mathfrak{p}} = 0 &\iff (\sigma(\omega_{i+1}, \mathcal{C}_{\omega_{i+1}}, f^{1-\varepsilon}))_{\mathfrak{p}} = (\omega_{i+1})_{\mathfrak{p}} \\ &\iff (\sigma(\omega_{i+1}, \widehat{\mathcal{C}_{\omega_{i+1}}}, f^{1-\varepsilon}))_{\mathfrak{p}} = \widehat{(\omega_{i+1})_{\mathfrak{p}}} \\ &\iff \operatorname{fit}^{\widehat{R_{\mathfrak{p}}}}_{i+1}(f) = 1 \\ &\iff F : H^i_{\mathfrak{p}}(\widehat{R_{\mathfrak{p}}}/f\widehat{R_{\mathfrak{p}}}) \to H^i_{\mathfrak{p}}(\widehat{R_{\mathfrak{p}}}/f\widehat{R_{\mathfrak{p}}}) \text{ is injective} \end{aligned}$$

where the completion is taken with respect to the maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$ of $R_{\mathfrak{p}}$. Note that the final equivalence follows from Proposition 5.10. Since

$$0 \neq H^i_{\mathfrak{p}}(R_{\mathfrak{p}}/fR_{\mathfrak{p}}) \cong H^i_{\mathfrak{p}}(\widehat{R_{\mathfrak{p}}/fR_{\mathfrak{p}}}) \cong H^i_{\mathfrak{p}}(\widehat{R_{\mathfrak{p}}}/f\widehat{R_{\mathfrak{p}}}),$$

we conclude that $(R/fR)_{\mathfrak{p}}$ is F-injective. □

We are now ready to prove $\mathfrak{m}$-adic constancy for the $\operatorname{fit}_i(f)$, under some assumptions on R/fR.

Theorem 5.12. *Let $(R, \mathfrak{m}, K)$ be an F-finite F-injective local ring of prime characteristic $p > 0$. Let $f \in R$ be a parameter such that R/fR is F-full and F-injective on the* *punctured spectrum. If $H^i_{\mathfrak{m}}(R) \neq 0$ for some integer $i \geqslant 0$, then there exists an integer $N \geqslant 0$ such that $\operatorname{fit}_i(f) = \operatorname{fit}_i(f+h)$ whenever $h \in \mathfrak{m}^N$.*

Proof. For such an i, let $\omega_i = (H^i_{\mathfrak{m}}(R))^\vee$, $\Phi = (F_{H^i_{\mathfrak{m}}(R)})^\vee \in \mathcal{C}^1_{\omega_i}$, and let $\mathcal{C}_i = R[\Phi_i]$. Since R/fR is F-full and F-injective on the punctured spectrum, we know that $(\omega_i/\sigma(\omega_i, \mathcal{C}_i, f^{1-\varepsilon}))_{\mathfrak{p}} = 0$ for every $\mathfrak{p} \subsetneq \mathfrak{m}$. We conclude that the R-module $\omega_i/\sigma(\omega_i, \mathcal{C}_i, f^{1-\varepsilon})$ has finite length. By Theorem 4.11, there exists $\geqslant 0$ such that $\sigma(\omega_i, \mathcal{C}_i, d^{1-\varepsilon}) = \sigma(\omega_i, \mathcal{C}_i, (f+h)^{1-\varepsilon})$ for all $h \in \mathfrak{m}^N$. As a consequence, $\operatorname{fit}_i(f) = \operatorname{fit}_i(f+h)$ for every $h \in \mathfrak{m}^N$. □

Our framework also extends known results on F-pure thresholds [HNBW18].

Corollary 5.13. *Let $(R, \mathfrak{m}, K)$ be a Gorenstein F-finite F-pure local ring of prime characteristic $p > 0$. Let R be a nonzerodivisor such that R/fR is F-pure on the punctured spectrum. Then there exists an integer $N > 0$ such that $\operatorname{fpt}(f) = \operatorname{fpt}(f+h)$ for every $h \in \mathfrak{m}^N$.*

Proof. This follows at once from Theorem 5.12 and the fact that F-pure thresholds coincide with F-injective thresholds for F-pure Gorenstein rings. □

5.3. ***F*-thresholds.** We now investigate the F-thresholds defined in Definition 3.1, which were initially defined for regular rings [MTW05], and then for general rings [HMTW08], provided certain limits exists. The existence of F-thresholds was later proved in full generality [DSNP18].

Our method is to interpret F-thresholds as σ-jumping numbers. We start with a result that uses ideas applied by Polstra and Tucker [PT18].

Proposition 5.14. *Let $(R, \mathfrak{m}, K)$ be a complete local Cohen-Macaulay F-finite domain of prime characteristic $p > 0$. If J is an ideal of R generated by a full system of parameters, then there exists an integer $N > 0$ with the following property: For all $e > 0$, if $\varphi(x) \in J^N$ for all $\varphi \in \operatorname{Hom}_R(F^e_* R, R)$, then $x \in J^{[p^e]}$.*

Proof. Let $d = \dim R$, and suppose that J is generated by the system of parameters $x_1, \ldots, x_d$. Let k be a coefficient field of R, and let $A = k[\![x_1, \ldots, x_d]\!]$ be a Noether normalization of R. Since R is Cohen-Macaulay, the A-module $F^e_* R$ is free. Fix $x \in R \smallsetminus J^{[p^e]}$. Since $\mathfrak{m}_A R = J$, we have that $x \notin (\mathfrak{m}_A R)^{[p^e]}$ or, equivalently, $x \notin \mathfrak{m}_A F^e_* R$. Therefore, x is part of a basis for $F^e_* R$. There is an A-linear map $\chi : F^e_* R \to A$ such that $\chi(x) = 1$. Let $\tau \in \operatorname{Hom}_A(R, A)$, and choose $y \neq 0$ such that $y \operatorname{Hom}_A(R, A) \subseteq \tau \cdot A$, which is possible because $\operatorname{Hom}_A(R, A)$ is a rank one torsion-free R-module. Furthermore, we can assume that $y \in A$. Choose $N > 0$ such that $y \notin \mathfrak{m}_A^N$. Consider the homomorphism $\varphi \in \operatorname{Hom}_R(F^e_* R, R)$, defined by the property $y\chi = \tau\varphi$, as in [PT18, Lemma 5.2]. Then $\tau(\varphi(x)) = y \cdot \chi(x) = y \notin J^N$. Therefore $\varphi(x) \notin \mathfrak{m}_A^N R = J^N$, otherwise $\tau(\varphi(x)) \in \mathfrak{m}_A^N$, by A-linearity of τ. □

An immediate consequence of Proposition 5.14 is the existence of an integer $N \geqslant 0$ for which $I_e(J^N) \subseteq J^{[p^e]}$ for all $e \geqslant 0$. In fact, we have equality when $N = 1$.

Corollary 5.15. *Let $(R,\mathfrak{m},K)$ be a complete local Gorenstein F-finite domain of prime characteristic $p>0$. If J is an ideal of R generated by a full system of parameters, then $I_e(J)=J^{[p^e]}$ for all $e\geqslant 0$. In particular $\mathrm{ct}^J_{\mathcal{C}_R}(\mathfrak{a})=c^J(\mathfrak{a})$, where $\mathcal{C}_R$ denotes the full Cartier algebra on R. Moreover, if $\mathcal{C}^e(\mathfrak{a})=R$ for some $e\geqslant 0$ (for instance, if R is F-rational), then $c^J(\mathfrak{a})$ is a rational number.*

Proof. With the notation from the proof of Proposition 5.14, observe that $\mathrm{Hom}_A(R,A)$ is a canonical module of R. Therefore, we can choose τ to be a generator of this module, and $y=N=1$. This shows that $I_e(J)\subseteq J^{[p^e]}$, while the other containment always holds. The equality $I_e(J)=J^{[p^e]}$ for all $e\geqslant 0$ immediately yields the equality $\mathrm{ct}^J_{\mathcal{C}}(\mathfrak{a})=c^J(\mathfrak{a})$. Finally, under the additional assumption that $\mathcal{C}^e(\mathfrak{a})=R$ for some $e\geqslant 0$, the rationality of $\mathrm{ct}^J_{\mathcal{C}}(\mathfrak{a})$ follows from Theorems 3.6 and 4.3. Observe that $\mathrm{fpt}(\mathfrak{a})>0$ when R is F-rational and Gorenstein. Thus, the first jumping number of $(R,\mathcal{C},\mathfrak{a})$ is positive, so that $\mathcal{C}^e(\mathfrak{a})=R$ for $e\gg 0$ by Lemma 2.19. ☐

Theorem 5.16. *Let $(R,\mathfrak{m},K)$ be a complete local F-finite Gorenstein domain of prime characteristic $p>0$, and let $\mathcal{C}$ denote the full Cartier algebra on R. Suppose that a nonzero element $f\in\mathfrak{m}$ satisfies the following properties: $\mathcal{C}^e(fR)=R$ for some $e\geqslant 0$, and R/fR is F-pure on the punctured spectrum. Then given an ideal J of R generated by a full system of parameters, there exists an integer $N>0$ such that $c^J(f)=c^J(f+h)$ for every $h\in\mathfrak{m}^N$.*

Proof. Note that R/fR is Cohen-Macaulay and, in particular, it is F-full [MQ18, Remark 2.4 (3)]. Theorem 4.11 ensures the existence of an integer N_1 such that if $h\in\mathfrak{m}^{N_1}$, then $\sigma(R,\mathcal{C},f^t)=\sigma(R,\mathcal{C},(f+h)^t)$ for all $t\in(0,1)$. Let $\alpha=\mathrm{ct}^J_{\mathcal{C}_R}(f)$. If $\alpha<1$, the result follows by taking $N=N_1$, since

$$c^J(f)=\mathrm{ct}^J_{\mathcal{C}}(f)=\mathrm{ct}^J_{\mathcal{C}}(f+h)=c^J(f+h)$$

by Corollary 5.15.

Now suppose that $\alpha\geqslant 1$. Fix N_2 for which $\mathfrak{m}^{N_2}\subseteq J$, and set $N=\max\{N_1,N_2\}$. Moreover, let $d=\lceil\alpha-1\rceil$ and $\beta=\alpha-d$. Given $h\in\mathfrak{m}^N$, write $(f+h)^d=f^d+g$ for some $g\in J$. By Proposition 2.20, for all integers $0\leqslant i\leqslant\beta$, we have that

$$\begin{aligned}\sigma(R,\mathcal{C},(f+h)^{d+i})&=(f+h)^d\sigma(R,\mathcal{C},(f+h)^i)\\&=(f^d+g)\sigma(R,\mathcal{C},(f+h)^i)=(f^d+g)\sigma(R,\mathcal{C},f^i)\end{aligned}$$

Since $g\in J$, we have that $(f^d+g)\sigma(R,\mathcal{C},f^i)\subseteq J$ if and only if $f^d\sigma(R,\mathcal{C},f^i)=\sigma(R,\mathcal{C},f^{d+i})\subseteq J$. The proof of Theorem 3.6 shows that α is the supremum over real numbers $t\geqslant 0$ for which $\sigma(R,\mathcal{C},f^t)\nsubseteq J\}$. It follows that $\alpha=\mathrm{ct}^J_{\mathcal{C}}(f+h)$, and proceeding as we did when $\alpha<1$ completes the proof. ☐

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