Midterm 2 Conceptual Review

Math 791, Spring 2019

- Midterm 2 is in class on **Thursday, April 11**.
- The exam is **cumulative**, but will focus on the material covered since the first midterm, which is focused on the related topics of **polynomials** and **fields**, and is primarily contained in:

$$\S\S 9.1 - 9.4,\ 7.5,\ 13.1 - 13.2,\ \text{and } 13.4.$$

  Consult me, or see the Daily Update, for a brief rundown on the particular topics we covered.
- The best preparation is to **practice, practice, practice** working and re-working problems. This includes **book problems** and **quiz problems**.
- Check out the **Daily Update** as a backup to check whether you understand all definitions and statements covered in class:
  - **For each definition**, try constructing your own example and counterexample.
  - **For each statement**, identify a realization of the result, and determine why each of its hypotheses is necessary.
- Check out **extended office hours** posted on the course website.

Polynomials (§§9.1 − 9.4)

- **Concepts**:
  - Polynomial ring in several variables
  - Polynomial rings (in one variable) over a field:
    - *Unique Factorization*
    - *Division Algorithm*, *Bézout's theorem*, and *Euclidean Algorithm*
    - this ring is a principal ideal domain
  - Irreducibility criteria:
    - *Root theorem*
    - A polynomial over a field of degree 2 or 3 is reducible $\iff$ it has a root
    - *Gauss' lemma*
    - *Rational root test*
    - *Eisenstein's criterion*
- **Goals**: Besides proving statements, and deducing conclusions, about polynomials and polynomial rings, be prepared to:
  - ☐ Apply the *Division Algorithm* for polynomials
  - ☐ Apply the *Euclidean Algorithm* to find a principal generator for an ideal
  - ☐ Apply root criteria to decide whether a polynomial of degree 2 or 3 is irreducible
  - ☐ Apply Gauss' lemma to determine irreducibility over $\mathbb{Q}$
  - ☐ Apply root criteria to decide whether a polynomial of degree 2 or 3 is irreducible
  - ☐ Apply Eisenstein's criterion over $\mathbb{Z}$, and over other rings
  - ☐ Apply irreducibility criteria in combination (and using other techniques such as shifting $x \mapsto x + a$) to determine irreducibility

- **Homework problems**: §9.1: 5; §9.2: 1, 2, 5, 6, 8 - 10; §9.3: 2; §9.4: 1 - 6, 8, 11 - 14, 16, 20
- **Additional problems**:
  - Find a principal generator for the following ideals in $\mathbb{Q}[x]$:

$$(x^3 - 6x + 7, x + 4), (x^2 - 1, 2x^7 - 4x^5 + 2), (x^3 - 1, x^7)$$

  - Determine whether the following polynomials are irreducible:

$$x^2 + 1 \in \mathbb{F}_{19}[x] \text{ and } x^4 + 2x^2 + 2, x^5 - 4x + 11, 5x^4 - 7x + 7, x^3 + 3x + 2 \in \mathbb{Q}[x]$$

  - Show that there exist infinitely many a for which $x^7 + 15x^2 - 30x + a$ is irreducible in $\mathbb{Q}[x]$.
  - Show that if F is a field and $f(x) \in F[x]$ has degree at least 1, then $f(x)$ is irreducible if and only if $f(ax + b)$ is irreducible for any $a, b \in F$, $a \neq 0$.

Fields $(\S 7.5, 13.1 - 13.2, 13.4)$

- **Concepts**:
  - The characteristic of a ring, and the prime subfield of a field
  - Field of fractions of a domain
  - Field K constructed from an irreducible $f \in F[x]$, F a field, containing a root of f:
    - Basis for K as a vector space over F, concrete description of the elements in K
  - Field extensions:
    - Degree/index, simple extension/primitive element, finitely generated extension
    - Algebraic/transcendental element/extension, algebraic closure, splitting field
    - Minimal polynomial of an algebraic element
- **Goals**: Besides proving statements, and deducing conclusions, about fields and field extensions:

  - ☐ Answer concrete questions about $F[x]/(f)$
  - ☐ Determine the degree of a field extension
  - ☐ Show that a given element is algebraic over a given field
  - ☐ Find a minimal polynomial
  - ☐ Determine a splitting field

- **Homework problems**: §7.5: 3, 4; §13.1: 1 - 4; §13.2: 1 - 5, 7, 8, 13, 14; §13.4: 1, 3, 4
- **Additional problems**:
  - Prove the associative law for addition for the field of fractions of a domain.
  - If a field F has prime characteristic p, show that $(a + b)^p = a^p + b^p$ for all $a, b \in F$.
  - Prove that $\cos(1°)$ is algebraic over $\mathbb{Q}$.
  - If $a \in \mathbb{C}$ is such that $p(a) = 0$, where $p(x) = x^5 + \sqrt{2}x^3 + \sqrt{5}x^2 + \sqrt{7}x + \sqrt{11}$, show that a is algebraic over $\mathbb{Q}$ of degree at most 80.
  - Given an example of $a, b \in \mathbb{C}$ that are algebraic of degrees 2 and 3, respectively, over $\mathbb{Q}$, but ab has degree less than 6. *Hint*: What is the degree over $\mathbb{Q}$ of a *third root of unity*?