Final Exam Conceptual Review

Math 791, Spring 2019

- The Final Exam is on **Wednesday, May 15** from **7:30 - 10 am**.
- The exam is **cumulative**, but will have a slight emphasis on the material covered since the second midterm, which is focused on group theory. The Conceptual Reviews for Midterms 1 and 2 are available on the course website.
  - Before Midterm 1, we covered material focused on **rings**: $0.3,\ 7.1-7.4,\ 7.5,$ and $9.1-9.3$.
  - Between Midterms 1 and 2, we covered material on **polynomial rings** and **fields** (with some overlap in sections): $7.5,\ 9.1-9.4,\ 13.1-13.2,$ and 13.4.
  - After Midterm 2, we studied **groups**: $1.1-1.6,\ 2.1-2.2,\ 2.4-2.5,\ 3.1-3.3,\ 4.5,$ and 5.2.

  Consult me, or see the Daily Update, for the particular topics that we covered in each section.
- The best preparation is to **practice, practice, practice** working and re-working problems. This includes **book problems** and **quiz problems**.
- Check out the **Daily Update** to check whether you understand all definitions and statements:
  - **For each definition**, try constructing your own example and counterexample.
  - **For each statement**, identify a realization of the result, and determine why each of its hypotheses is necessary.
- Check out **extended office hours** that will be posted on the course website.

Basics on groups $(1.1-1.4, 1.6)$

- **Concepts**: Group, inverse, identity, order of a group, order of an element in a group, group homomorphism/isomorphism, fundamental examples: *dihedral*, *symmetric*, and *matrix groups*.
- **Goals**: Prove statements and make conclusions about groups with certain properties, or for specific examples of groups. Construct homomorphisms and isomorphisms between groups.
- **Problems**: *Homework*: 1.1: #6, 7, 13, 14, 20, 21, 25; 1.6: #5, 17, 18; Investigation Module 2

1. Prove that if G is a group and $x^{-1} = x$ for all $x \in G$, then G is abelian.
2. Show that any group of order at most 4 is abelian. *Harder*: Do the same for those of order 5.
3. If G is finite a group with even order, prove that $x^{-1} = x$ for some $x \neq 1$ in G.
4. In a finite group, prove that every element has a finite order.
5. Show that if n is odd, then the only element $x \in D_{2n}$ such that $xy = yx$ for all $y \in D_{2n}$ is $x = 1$. On the other hand, if n is even, prove that there exists $x \neq 1$ with this property.

Subgroups $(2.1-2.3)$

- **Concepts**: Subgroup, (finite) subgroup criterion, centralizer, normalizer, cyclic subgroup.
- **Goals**: Prove statements about subgroups. Determine whether a subset of a given group is a subgroup. Compute a centralizer or normalizer.
- **Problems**: *Homework*: 2.1: #4–7; 2.2: #2, 6; 2.3: #6, 7

1. If G is abelian, prove that the set $\{x \in G \mid x^2 = 1\}$ is a subgroup of H. Exhibit an example in which G is not abelian, and this set is not a subgroup.

2. Prove the subgroup criterion.
3. If G is a group and $x, a \in G$, prove that $C_G(x^{-1}ax) = x^{-1}C_G(a)x$.

Cyclic groups (2.3)

- **Concepts**: Cyclic group, cyclic subgroup, generator of a cyclic group.
- **Goals**: Prove statements about cyclic groups and cyclic subgroups. Determine whether a given group or subgroup is cyclic. Describe the distinct elements of a cyclic group. Classify the subgroups of a cyclic group. Find all generators of a cyclic group. Apply the fact that any two cyclic groups of the same order are isomorphic.
- **Problems**: *Homework*: 2.3: #2, 6, 7, 12–15

1. Prove that every subgroup of a cyclic group is cyclic.
2. Prove that if G is a group, and has no subgroups besides 1 and G, then G must be cyclic. *Harder*: Show that in this case, G must also have prime order.

Normal subgroups and quotient groups (3.1)

- **Concepts**: Kernel of a group homomorphism, group of fibers of a group homomorphism, left/right coset of a subgroup in a group, representative of a coset, conjugate of a subgroup by a group element, normal subgroup, quotient group modulo a normal subgroup.
- **Goals**: Work with the group of fibers of a group homomorphism. Prove and apply properties of group homomorphisms. Investigate the left or right cosets of a subgroup in a group. Prove two cosets coincide or are not the same. Determine whether a subgroup is normal using various criteria. Prove general properties about cosets, normal subgroups, and quotient groups. Work in a quotient group modulo a normal subgroup.
- **Problems**: *Homework*: 3.1: #3, 14, 17, 20, 34, 35

1. If $\varphi : G \to H$ is a surjective homomorphism and G is abelian, prove that H is also abelian.
2. In the setup of (1), prove that if $N \trianglelefteq G$, then $\varphi(N) \trianglelefteq H$.
3. Prove that if N, M are normal subgroups of a group G, then $MN \trianglelefteq G$.
4. Prove that a quotient of a cyclic group modulo a normal subgroup is again cyclic.
5. If G is a group, $N \trianglelefteq G$, and G/N is abelian, prove that $aba^{-1}b^{-1} \in N$ for all $a, b \in G$.

Lagrange's theorem (3.2)

- **Concepts**: order of a group/subgroup, left/right coset of a subgroup in a group, index of a subgroup in a group, *Lagrange's theorem*, *Cauchy's theorem*, preliminary version of *Sylow's theorem*, product of subgroups, any group of prime order is cyclic.
- **Goals**: Apply Lagrange's theorem to make conclusions about the order of a subgroup, the number of left cosets (i.e., the index) of a subgroup in a group, and the order of a quotient group. Make conclusions about the orders of elements in a finite group.
- **Problems**: *Homework*: 3.2: #2, 8, 11, 16

1. If H is a subgroup of a group G and $x, y \in G$, assume that $aH \neq bH$. Must the right cosets Ha and Hb also be distinct?

2. Exhibit a bijection between the left cosets and right cosets of a subgroup in a group. (It is *not* the obvious map $gH \leftrightarrow Hg$!)
3. For $H \leq G$, if $g \in G$ and $m > 0$ is smallest for which $g^m \in H$, prove that m divides $|g|$.
4. Use Lagrange's theorem to prove *Fermat's little theorem*: If p is prime and $p \nmid a$, then $a^{p-1} \equiv 1 \bmod p$.
5. Exhibit an example of subgroups H, K of a group G for which HK is not a subgroup of G.
6. Prove that if H and K are subgroups of an abelian group, then HK is a subgroup of order mn if and only if m and n are relatively prime.

The isomorphism theorems for groups (3.3)

- **Concepts**: The *first, second/diamond, third* and *fourth/lattice isomorphism theorems* for group, lattice of subgroups of a group, correspondence between subgroups of a group and a quotient of this group by a normal subgroup.
- **Goals**: Prove (parts of)the isomorphism theorems. Use the first isomorphism theorem to construct isomorphisms between groups. Find the index of the kernel of a group homomorphism from a group, in this group. Apply the second/diamond isomorphism theorem to compare quotient groups. Use the third isomorphism theorem to understand a quotient of a quotient group. Apply the fourth/lattice isomorphism theorem to understand the structure of and relationship between the subgroups of a quotient group.
- **Problems**: *Homework*: 3.3: #1–4, 7

1. Let G be the group of real-valued functions on the unit interval $[0, 1]$, under the operation of function addition. If $N = \{f \in G \mid f(1/4) = 0\}$, prove that G/N is isomorphic to $\mathbb{R}$.
2. If $N = \{-1, 1\} \leq \mathbb{R}^\times$, prove that $\mathbb{R}^\times/N$ is isomorphic to $(\mathbb{R}^+, \cdot)$.
3. If G and H are groups, prove that $N = \{(g, 1) \mid g \in G\}$ is a normal subgroup of $G \times H$. Then prove that $N \cong G$, and $(G \times H)/N \cong H$.

Sylow's theorem (4.5)

- **Concepts**: p-group/subgroup, Sylow p-subgroup, *Sylow's theorem*, simple group.
- **Goals**: Apply Sylow's theorem to understand the structure of finite groups. Determine the number of Sylow p-subgroups of a group. Prove that a given group, or group with certain properties, is simple or not simple. Characterize groups with a certain order.
- **Problems**: *Homework*: 4.5: #1, 4, 5, 13, 16, 30

1. If P is a Sylow p-subgroup of a finite group G, then P is also a Sylow p-subgroup of $N_G(P)$, and is the only Sylow p-subgroup of $N_G(P)$.
2. Show that a group of order 108 has a normal subgroup of order 9 or 27.

The fundamental theorem of finite abelian groups (5.2)

- **Concepts**: *Fundamental theorem of finite abelian groups.*
- **Goals**: Determine/classify all isomorphism classes of finite abelian groups with a given order. Study the subgroups of finite abelian groups.
- **Problems**: *Homework*: 5.2: #1, 4, 9

1. Suppose that G is a finite abelian group of order p^n for a prime p. Prove that if $a \in G$ has maximal order among all elements of G, then $g^{|a|} = 1$ for all $g \in G$.