Midterm 1 Conceptual Review

MATH 558, Fall 2020

Introductory Modern Algebra

- Midterm 1 will be taken online during our class period on **Thursday, September 24**.
- Our exam covers course material related to the following sections in our text: 2A–2C and 3A–3E.
- You may use a **scientific calculator**, but it should only be used to check arithmetic. In particular, you must use methods from class wherever applicable, and show your work.
- The best preparation is to **practice, practice, practice** working, and re-working problems. This includes **homework**, **quiz**, and **investigation module** problems.

Mathematical Induction and Well-ordering

Concepts: The Principle of Mathematical Induction, the Principle of Complete Induction, base case, inductive step, inductive hypothesis, Well-ordering Principle.

Goals: Apply PMI or PCI to prove statements about infinitely many integers, choosing the correct method, checking the base case, stating the inductive hypothesis accurately, and carrying out the inductive step. Apply WOP to prove statements about integers.

Assignments: Chapter 2: #14, 26, 30–32, 34, 35; Quiz 1; Quiz 2; Investigation Module 1

Videos:
- Intro to mathematical induction
- Proving an inequality via induction
- Proving a geometric property via induction
- Proving a divisibility property via induction
- Intro to complete induction
- Proving divisibility by a prime via complete induction
- Proving a property about a sequence via complete induction
- The equivalence of induction and complete induction
- The Well-ordering Principle
- Proving statements via the Well-ordering

Additional practice problems:

(1) Prove that for all integers $n \geq 1$, $\frac{d}{dx}x^n = nx^{n-1}$.

(2) Prove that for all positive integers n,

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

(3) Prove that $2^{2n} \geq n^4$ for all integers $n \geq 4$.

(4) Prove that for every integer $n \geq 1$, $5 \mid (3^{4n} - 1)$.

(5) Prove that any integer $n \geq 2$ factors into a product of primes (where we consider a prime itself to be a product of one prime).

(6) Use well-ordering to prove that any two integers have a least common multiple.

(7) Prove the existence of q, r in the Division Algorithm using well-ordering.

Greatest Common Divisors

Concepts: Divisor/divides, multiple, greatest common divisor, Division Algorithm, Euclidean Algorithm, Extended Euclidean Algorithm, Bézout's theorem.

Goals: Prove statements about integers in terms of divisibility, apply the Division algorithm to make conclusions, find the greatest common divisor of a pair of integers using the Euclidean algorithm, find integers satisfying Bézout's identity using the Extended Euclidean Algorithm, use Bézout's theorem to prove statements about integers.

Assignments: Chapter 3: #4, 5, 23, 26–28, 35, 39, 40, 43, 44; Quiz 3; Investigation Module 2

Videos:
- Intro to divisibility
- The Division Algorithm
- Greatest common divisors
- More on divisibility and greatest common divisors
- Bézout's theorem
- Example of the Euclidean Algorithm and its Extended version

Additional practice problems:

(1) Show that any two consecutive integers are relatively prime. Then show that any two consecutive odd integers are relatively prime.

(2) Show that of any three consecutive integers, exactly one is divisible by 3.

(3) Find $(112345, 112354)$, and an associated Bézout identity.

(4) Find a counterexample to the following: Given $a, b, c \in \mathbb{Z}$, if $a \mid bc$ and $a \nmid b$, then $a \mid c$. Then show that if a is an integer for which, whenever $a|bc$ and $a \nmid b$, then necessarily $a|c$, then a must be prime.

(5) Given $a, b, c \in \mathbb{Z}$, prove that if $(a, c) = d$ and $(b, c) = 1$, then $(ab, c) = d$. (Make sure to show that this GCD equals d, not just that it is at least d!)

(6) Prove that if $a, b, r, s \in \mathbb{Z}$, and $d = (a, b)$, if $ra + sb = d$, then $(r, s) = 1$.

(7) Prove that if a, b, n are integers for which $n|a$ and $n|b$, then $n|(a, b)$.

Linear Diophantine Equations

Concepts: Linear Diophantine equation, solutions to LDE.

Goals: Determine whether a LDE has a solution, find its infinitely many solutions if equation has a solution.

Assignments: Chapter 3: #57, 64

Videos:
- Homogeneous linear Diophantine equations
- Solving linear Diophantine equations

Additional practice problems:

(1) Find the solution with the smallest $x > 0$ to the equation $1001x + 143y = 0$.

(2) Find all solutions to $44x + 55y = 8$.

(3) Find all solutions to the following: (i) $242x + 1879y = 66$, (ii) $327x + 870y = 66$, (iii) $327x + 870y = 56$.

(4) If a solution exists, find the solution with the smallest possible $x \geq 0$: (i) $133x + 203y = 38$, (ii) $133x + 203y = 40$, (iii) $133x + 203y = 42$. (iv) $133x + 203y = 44$.

(5) For each equation from (4), find the solution (if one exists) with smallest possible $y \geq 0$.