Daily Update

Math 500: Intermediate Analysis, Spring 2017

This is a brief summary of what was covered in lecture; please `email me` if you find a typo, and bring question to office hours and class.

Lecture 28: Thursday, May 4. We started class by discussing series of functions $\sum_{k=1}^{\infty} f_k(x)$, and the function $g(x)$ that they define on input values x for which the series converges. We defined what it means for such a series of functions to be **uniformly convergent** to $g(x)$. We proved that if each $f_k(x)$ is continuous on an interval, and the series is uniformly convergent to g, then g must also be continuous. Next, we stated and gave the ideas behind the **Weierstrass M-test**, giving us one criterion that a series of functions can satisfy to ensure that the series converges uniformly to g. We discussed an example of applying these two results.

Next, we recalled the definition of a **power series** $\sum_{k=0}^{\infty} c_k(x-a)^k$ centered at $x = a$, and proved that

$$R = \frac{1}{\limsup |c_k|^{1/k}}$$

is the **radius of convergence** in the sense that we remember from calculus class, and in even a stronger (uniform) sense! We found the interval of convergence in some examples using this definition of R.

Finally, we discussed the fact that power series can be integrated and differentiated term-by-term, and the new series have the same radius of convergence as the original one. We finished the lecture with the statement of Taylor's formula, and briefly discussed its significance

Finally, we reviewed some of the kinds of problems that are likely to show up on the Final Exam.

Lecture 27: Tuesday, May 2. We started class by recalling the Alternating Series Test; we checked that both hypotheses are necessary by exhibiting examples where the conclusion fails after removing a hypothesis. This test gives us nice examples of absolutely convergent series, like

$$\sum_{k=1}^{\infty}(-1)^{k+1} \cdot \frac{1}{k^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots ,$$

and conditonally convergent series, like

$$\sum_{k=1}^{\infty}(-1)^{k+1} \cdot \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

We defined the **rearrangement** of a series, and then stated the following theorems:

1. **Theorem.** If an absolutely series converges to a real number s, then every rearrangement of the series also converges to s.

2. **Theorem.** Given a conditionally convergent series, for any extended real number L, there exists a rearrangement of the series that converges to L.

We gave the idea of how to prove (2) by considering the alternating harmonic series $\sum_{k=1}^{\infty} (-1)^{k+1} \cdot \frac{1}{k}$, which converges to $\ln(2)$. First, we found a rearrangement that converges to $\frac{3}{2}\ln(2)$, and then we described methods of how to create a rearrangement that converges to 0, and another that diverges to ∞! We found the first terms of these rearrangements. Important to the method is that

- The terms approach zero,
- There are infinitely many positive, and negative, terms, and
- The series of all positive, and all negative, terms diverge.

Finally we proved (1) using the definition of convergence, and absolute convergence, of series.

Lecture 26: Thursday, April 27. Today, we proved several theorems on infinite series that we recall from calculus using techniques we have learned throughout our course, including methods involving limits, limit suprema, and integrals.

We started by proving our generalized **Comparison Theorem** for infinite series, using the fact that a sequence is Cauchy if and only if it converges. We did an example of a "non-traditional" use of this comparison, where we applied L'Hôpital's rule, and the definition of a limit.

Next, we stated and proved the **Integral Test**. We noticed that the ***p*-test** comes for free from the integral test. We stated a stronger version of the **Root Test** than is given in calculus class, as well as the **Ratio Test**, and showed how our new version of the former can be more former than the latter.

Next time, we will turn to some facts about series that are new, and probably surprising!

Lecture 25: Tuesday, April 25. So far, we have only defined integrals of functions that are bounded on a closed, bounded interval. We began class with a discussion on *improper integrals*, integrals where either the interval is not bounded, or the function is not bounded on the given interval To define such an integral, we use limits. Each time, we (possibly) need to first break up the integral into an improper integral that is improper only because of *one* of its two endpoints. Next, we replace the "problem" endpoint and take a limit.

For example, if f is bounded on an interval $[a, \infty)$ for a real number a, the integral $\int_a^\infty f(x)\,dx$ is still improper since the interval $[a, \infty)$ is not bounded. We define

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

Likewise, we can define $\int_{-\infty}^b f(x)\,dx$ if f is bounded on $(\infty, b]$ via a limit, and we define $\int_{-\infty}^\infty f(x)\,dx$ by writing it as

$$\int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

for some a, and recalling the definition of each improper integral in the sum (in terms of a limit).

We proved that the improper integral $\int_1^\infty \frac{1}{x^p}\,dx$ converges to $\frac{1}{1-p}$ if $p > 1$, and diverges if $p \leq 1$. For homework, you will verify for which values of p the improper integral $\int_0^1 \frac{1}{x^p}\,dx$ converges and diverges.

Notice that this last integral is improper since it is not bounded on the interval $(0,1]$ (notice that $\frac{1}{x}$ is not defined at $x = 0$). We define this type of improper integral as a limit, if it exists:

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{a\to 0^+} \frac{1}{x^p}\,dx;$$

if this limit does not exist, we again say that the integral diverges. We pushed these ideas further, giving examples of how to turn several improper integrals into sums of improper integrals with only one endpoint that makes each improper; then we turned each of these integrals into limits of integrals on closed, bounded sets (proper integrals).

Next, we defined an **(infinite) series** as a formal sum of real numbers a_k:

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \ldots.$$

Associated to each series is its **sequence of partial sums**:

$$s_n = a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k.$$

We say that the infinite sequence **converges** and **equals** s if $\lim_{n\to\infty} s_n$ converges, and its limit is the real number s.

We gave several examples. We first showed that the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ converges and equals 1. However, we were not able to find a formula for the n-th partial sum of the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ in order to investigate whether the sequence of partial sums has a limit. We agreed that the series $\sum_{k=1}^{\infty} \frac{k+5}{100k+3}$ should diverge since the terms approach $\frac{1}{100}$, and make this formal by proving a theorem, sometimes called the **term test**: a series converges if and only if the limit of its terms limit to zero. However, the converse statement does not hold: by grouping terms and creating a new series from it, we showed that the series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

We next considered geometric series, those of the form:

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \ldots,$$

where $a \neq 0$ and r are real numbers. We proved that such a series converges if and only if $|r| < 1$; in this case, the series converges to $\frac{a}{1-r}$.

Next, we showed that if $a_k \geq 0$ for all k, then $\sum a_k$ converges if and only if the sequence of partial sums is bounded above, using the Monotonic Convergence Theorem.

Finally, we stated a strengthening of the **Comparison Test**: Given series $\sum a_k$ and $\sum b_k$, where all $b_k \geq 0$, if there exist $K, M$for which

$$|a_k| \leq M b_k \text{ for all } n \geq K,$$

then if $\sum b_k$ converges, so does $\sum a_k$.

Lecture 24: Thursday, April 20. Today was driven by the motivating question: **What exactly is the number e?**

We first noticed that the "rule" that $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ works for all rational numbers except for $n = 1$.

We then used the Second FTC to construct a function called the "natural logarithm," whose domain is all positive numbers, that is an antiderivative of $\frac{1}{x}$ on this domain:

$$\ln(x) = \int_1^x \frac{1}{t}\,dt.$$

We noticed that $\ln(1) = 0$ from the definition. Every antiderivative of $\frac{1}{x}$ on $(0,\infty)$ differs by a constant from $\ln(x)$, so this is the unique antiderivative with value zero at $x = 1$.

From here, we first investigated the antiderivative(s) of $\frac{1}{x}$ on the interval $(-\infty, 0)$; as this function is continuous here, it has an antiderivative. We calculated (using substitution for integrals) that $\ln(-x) = \ln|x|$ (which makes sense, since $-x = |x|$ is positive if x is negative) is an antiderivative, so functions of the form $\ln|x| + C$ are all antiderivatives of $\frac{1}{x}$ for $x \neq 0$.

We proved that $\ln(ab) = \ln(a)+\ln(b)$ for all $a, b > 0$ using the fact that $\ln(x)$ is an antiderivative of $\frac{1}{x}$. As exercises, you will also show that for all $a, b > 0$ and $r \in \mathbb{Q}$, that

- $\ln(a/b) = \ln(a) - \ln(b)$, and
- $\ln(a^r) = r\ln(a)$.

We proved that $\ln(x)$ is strictly increasing on $(0,\infty)$ by finding its derivative using the Second FTC. We also showed that $\lim\limits_{x\to\infty} \ln(x) = \infty$ and $\lim\limits_{x\to 0^+} \ln(x) = -\infty$ using the properties of natural logarithm just established, and a straightforward argument applying the definition of a limit.

These facts show that $\ln(x)$ has domain $(0,\infty)$, and its set of output values is $\mathbb{R}$! Then, since it is strictly increasing, it is strictly monotonic, and therefore has an inverse. We define the function $\exp(x)$ as its inverse function, so that $\exp(x)$ has domain $\mathbb{R}$, and its set of output values is $(0,\infty)$.

We showed that $\exp(x)$ is its own derivative using the fact that its inverse is $\ln(x)$. Next, we showed some properties (analogous to those for $\ln(x)$): for all $a, b > 0$ and $r \in \mathbb{Q}$,

- $\exp(a + b) = \exp(a)\exp(b)$, and
- $\exp(ra) = (\exp(a))^r$.

We think (know?) that $\exp(x)$ should "equal" e^x, so these would translate to $e^{a+b} = e^a e^b$ and $e^{ra} = (e^a)^r$.

From here, we define the number e precisely!

$$e = \exp(1).$$

In other words, since $\exp(x)$ and $\ln(x)$ are inverses, e is the unique number for which $\ln(e) = 1$. We checked that this means $e^r = \exp(r)$ for any rational number r, and we can then define e^x for x a real number:

$$e^x = \exp(x).$$

From here, we noticed that it is easy to define a^x and $\log_a(x)$ for $a > 0$, so that these are inverses, and the typical exponent rules work.

Finally, we started considering *improper integrals*, integrals where either the domain is unbounded, or the function in question is unbounded on its domain. If f is integrable on each interval of the form $[a, s]$, then we define

$$\int_a^\infty f(x)\,dx = \lim_{s\to\infty} \int_a^s f(x)\,dx.$$

We used this definition to find that $\int_0^\infty e^{-x}\,dx = 1$.

We can define integrals of the form $\int_{-\infty}^b f(x)\,dx$ similarly. For next time, read (or remind yourself) about integrals where the function is unbounded on the interval under consideration.

Lecture 23: Tuesday, April 18. We started class by recalling the Second Fundamental Theorem of Calculus, and finishing its proof. We applied the theorem to find derivatives of functions that are compositions of the antiderivative F constructed in the Second FTC, and another function.

From here, we used the First and Second Fundamental Theorems to prove the substitution rule, and integration by parts, rules we take for granted!

Lecture 22: Thursday, April 13. We started class by recalling some of the important theorems on integrals that we proved last time, and the tools we used to prove them. Next, we stated two more: One on the integral of the absolute value of a function, and one on how integral behave with respect to subintervals; the second is as follows: Fix $c \in [a, b]$. Then a function f integrable on $[a, b]$ if and only if f is integrable on both $[a, c]$ and $[a, b]$, in which case

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Next, we turned to the most important theorems in analysis: The Fundamental Theorems of Calculus! We started by stating the **First Fundamental Theorem of Calculus (FTC)**: Given a function f for which:

- f is continuous on $[a, b]$,
- f is differentiable on (a, b), and
- f' is integrable on $[a, b]$,

then

$$\int_a^b f'(x) = f(b) - f(a).$$

From here, we first recified a strange part of this statement: If f is not differentiable at an endpoint a or b, we can still consider f' to be *integrable* on the closed interval $[a, b]$, since the area under the curve will be the same as if we "artificially" assign values of f' at a and/or b. We illustrated this with a picture, and we came back later in class to do a concrete example using an equation.

We proved the First FTC using the Mean Value Theorem, Riemann/upper/lower sums, and the squeeze theorem for limits.

Next, we considered the function

$$f(x) = \begin{cases} x^2\cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

We showed that $f(x)$ is continuous and differentiable everywhere, and also showed that $f'(x)$ is integrable on any closed interval, but it is *not* continuous at $x = 0$. This helps us see how functions whose derivatives are not continuous may satisfy our hypotheses in the First FTC.

From here, we stated the **Second FTC**: Suppose that f is integrable on $[b, c]$, and fix $a \in [b, c]$. Define a function

$$F(x) = \int_a^x f(t)\,dt$$

for $x \in [b, c]$. Then F is continuous on $[b, c]$. Moreover, for each point $x \in (b, c)$ for which f is continuous, F is differentiable at x, and

$$F'(x) = f(x).$$

We illustrated the function $F(x)$ using a graph, and then proved the first part of the theorem, the continuous on $[b, c]$. In fact, it is uniformly continuous there!

Lecture 21: Tuesday, April 11. We started class by recalling the following: If f is a bounded function on an interval $[a, b]$, and there exists a sequence of partitions $\{P_n\}$ on $[a, b]$ for which

$$\lim_{n\to infty} (U(f, P_n) - L(f, P_n)) = 0,$$

then f is integrable on $[a, b]$, and

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n),$$

and also equals the limit of any sequence of Riemann sums with respect to P_n.

From here, our goal for today was to investigate **what functions are integrable**.

We proved that any **monotonic function** on a closed, bounded interval $[a, b]$ is integrable on $[a, b]$. We also showed that any **continuous** function on $[a, b]$ is integrable on $[a, b]$. Each of these proofs involved finding limits of the difference between upper and lower sums!

Next, we investigated **linearity of the integral**: Given f, g integrable functions on $[a, b]$, and $c \in \mathbb{R}$,

1. cf is integrable on [a,b], and $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
2. $f + g$ is integrable on [a,b], and $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

We proved (1) using facts about suprema and infinima, and gave some ideas on how to show (2).

We noticed that as a corollary, if f is an integrable function on $[a, b]$, then

$$(\inf_{[a,b]} f)(b-a) \le \int_a^b f(x)\,dx \le (\sup_{[a,b]} f)(b-a).$$

We defined the **mean/average value** of an integrable function f on $[a, b]$ as

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

We proved, using the Mean Value Theorem, that if f is also continuous on $[a, b]$, then there exists $a \le c \le b$ for which $f(c)$ equals the mean value of f on $[a, b]$.

Lecture 20: Tuesday, April 4. Today, we recalled the definition of upper and lower sums corresponding to a partition of a bounded function on a closed interval. We noticed that this does not require the function to be continuous on the interval! We also recalled the definition of the upper and lower integrals, and showed that the former is always at least the latter. If the upper and lower integrals are equal, we define the **Riemann integral** of a bounded function f on an interval $[a, b]$ to be the common value $\underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx$, and denote it by

$$\int_a^b f(x)\,dx.$$

We proved that the Riemann integral of f on $[a, b]$ exists if and only if for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ for which

$$U(f, P) - L(f, P) < \varepsilon.$$

From this theorem, we can conclude that the Riemann integral exists if and only if there exists a sequence of partitions $\{P_n\}$ for which

$$\lim_{n\to\infty} (U(f, P_n) - L(f, P_n)) = 0.$$

In this case,

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} L(f, P_n),$$

and also equals the limit of any Riemann sum corresponding to the partitions $\{P_n\}$.

We found $\int_0^2 x^2\,dx$ using these methods (along with the given formula $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$), and almost finished finding $\int_0^1 e^x\,dx$ using the formula for the sum of a geometric series, and L'Hôpital's rule.

For Midterm 2, you will need to calculate the Riemann integral of a simple function f on a closed interval $[a, b]$, as in Example 5.1.9. For this, we can follow the following procedure:

1. Fix the simple partition of $[a, b]$ into n equal parts.
2. Find the corresponding upper and lower sums.
3. Show that the limit of the difference of these sums equals zero.
4. Find the Riemann integral as the limit of either the upper or lower sum.

Lecture 19: Thursday, March 30. Today, we defined a **partition** of a closed interval, and a **Riemann sum** associated to this partition. Note that this is more general that our typical calculus definition, as a partition need not split the interval into subintervals of equal length, and any arbitrary function value instead of a specified (left-hand, right-hand, midpoint) value. However, any Riemann sum still represents the sum of areas of rectangles (possibly with signs added) as we are used it.

We defined the **upper sum for a function f on a partition P**, $U(f, P)$ via replacing the function value in the definition of a Riemann sum with the supremum of all function values among input values in the subinterval in question. The **lower sum** $L(f, P)$ is defined analogously. We noticed that $L(f, P)$ less than or equal to any Riemann sum on an interval, which in turn is less than or equal to $U(f, P)$. We also computed an explicit example.

Next, we defined a **refinement** of a partition P as a partition Q containing all points of P; i.e., $P \subseteq Q$. We gave an argument for the following **theorem**: If f is a bounded function on a closed interval, and Q is a refinement of P, a partition on this interval, then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

Then we used this to prove an interesting **theroem**: Given any two partitions P and Q on a closed interval, if f is bounded on this interval, then

$$L(f, P) \leq U(f, Q).$$

We defined the **upper integral** and **lower integral** of f on a bounded interval $[a, b]$ as"

$$\overline{\int_a^b} f dx = \inf\{U(f, Q) \mid Q \textbf{ partition on } [a, b]\}$$
$$\underline{\int_a^b} f dx = \sup\{L(f, Q) \mid Q \textbf{ partition on } [a, b]\}$$

The theorem above says that every lower sum is less than or equal to every upper sum, so that every upper sum is an upper bound for the set of lower sums; therefore, $\overline{\int_a^b} f dx \leq U(f, P)$ for every partition P. We did not finish arguing that this means that $\overline{\int_a^b} f dx$ is less than or equal to the least upper bound of the set of all upper bounds; i.e.,

$$\underline{\int_a^b} f dx \leq \overline{\int_a^b} f dx.$$

We will use this to define the integral! Please note that §5.1 will be the last section covered on Midterm 2. You may want to read ahead in §5.1 to complete the homework, or wait on some until after Tuesday's lecture, which will include both discussion of the integral, and some review.

Lecture 18: Tuesday, March 28. We started class by proving that any two antiderivatives of a differentiable function on an open interval differ by a constant, a fact we use freely often in calculus! This was a direct application of the Mean Value Theorem.

Next, we stated a generalization of the MVT, **Cauchy's MVT**. Then we used this to prove **L'Hôpital's Rule**, which was involved, using three limits to construct "epsilon" values that work together, along with the triangle inequality.

We applied L'Hôpital's Rule to find several limits, including $\lim\limits_{x\to 1} \frac{\sin(\pi x)}{\ln x}$, $\lim\limits_{x\to\infty} \frac{e^x}{x^{100}}$ (where we needed to apply the rule 100 times), and the (in)famous $\lim\limits_{n\to\infty} 1 + \frac{p}{n}^n$, where $p \in \mathbb{R}$ is constant. Notice for the last limit, that each term agrees with the differentiable function $y = (1+\frac{p}{x})^x$ at the integer $x = n$. Then we can find $\lim\limits_{n\to\infty} \ln((1+\frac{p}{n})^n)$, and use this to find the original limit. We also found $\lim\limits_{x\to 0} x^x$ using a similar method.

Lecture 17: Thursday, March 16. Today, we recalled the definition, and pointed out again that we can define the derivative on an *open interval*, referring to some examples from last time.

We showed that function that is differentiable at a point must be continuous at that point, and gave a couple examples of functions that are continuous, but not differentiable.

We also stated some basic derivative rules that weren't covered last time, including the chain rule. We used the chain rule to derive the formula for the derivative of the inverse of a function, and looked at an example.

Next, we stated and proved the **Mean Value Theorem** (MVT), which essentially states that if f is a continuous function on a closed interval $[a, b]$ that is differentiable on (a, b), then for at least one point between a and b, the graph of f has tangent line parallel to the line joining $(a, f(a))$ and $(b, f(b))$ on the graph of f. This is a powerful and useful theorem!

This will help us find *all* antiderviatives of a given function!

Lecture 16: Tuesday, March 14. We started of by recalling what it means to say that $\lim\limits_{x\to a} f(x) = L$ and compared it with the definition of what it means for f to be continuous at a point $x = a$ to once again see that f is continuous at $x = a$ if and only if both of the following hold

1. a is in the domain of f.
2. $\lim\limits_{x\to a} f(x) = f(a)$.

Following this, we gave the following **definition**: If f is defined in an open interval containing the point a, then the *derivative* of f at $x = a$ is

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a},$$

which we later argued, after making the substitution $x = a + h$ is the same as the limit

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

We say that f is *differentiable at* a if the derivative exists at a, and *differentiable on an open interval* I if the derivative exists at every point $a \in I$.

We then argued that if $f'(a)$ exists, then $\lim\limits_{x\to a} f(x) = f(a)$ (that is, the numerator of the fractions above must go to zero in the limit). We also stressed the following: To verify that $f'(a)$ exists using $\varepsilon - \delta$ methods, we need to first have a candidate L for the limit, and then play the $\varepsilon - \delta$ game to prove that $\lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a} = L$, which is often burdensome and tricky. We explicitly did this to show that if $f(x) = x^2$, then f is differentiable on $\mathbb{R}$, and $f'(x) = 2x$.

Using the earlier facts that we reviewed about continuity, we were able to skip the $\varepsilon - \delta$ game to more and still show that if $f(x) = x^n$ with n a positive integer, then $f'(a) = na^{n-1}$. Using the definition of the derivative, we also recovered the well-known formula for the derivative of $f(x) = \sqrt{x}$. Notice that although the domain of $\sqrt{x}$ is $[0, \infty)$, it is differentiable only the open interval $(0, \infty)$ (compare this to the derivative formula!).

To illustrate how much we take for granted from calculus, we tried to use the definition of the derivative to explicitly show that the derivative of $\sin(x)$ is $\cos(x)$ (this involves some double-angle formulas for $\sin(x)$) and that the derivative of e^x is e^c (this involved a deeper understanding of what the constant e actually means!). After realizing these subtle points, we then started proving some basic formulas for derivatives. Using only the definition of the derivative, we proved the product rule $(fg)' = f'g + fg'$. We also used the product rule to prove the well-known quotient rule $(f/g)' = \frac{f'g - fg'}{g^2}$. We concluded the class with Quiz 7.

Lecture 15: Thursday, March 9. We started class by stating a theorem guaranteeing that a sequence of functions converges uniformly to the zero function, and then applied it to the sequence $\{\sin(nx)/n\}_{n=1}^{\infty}$, where the domain of each function is $\mathbb{R}$.

Next, we defined the limit of a function at a point (if it exists): Suppose that f is a function defined at all points on an open interval, except possibly at $a \in I$. Then we say that the limit of $f(x)$ as x approaches a equals L, and write

$$lim_{x \to a} f(x) = L,$$

if given any $\varepsilon > 0$, there exists $\delta > 0$ for which

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

We noticed that the condition $0 < |x - a|$ precisely means that $x \neq a$; this must be a requirement since $f(a)$ does not have meaning if a is not in the domain of f.

We also noticed that if f is defined at a, then if L is replaced with $f(a)$, then this is precisely the definition of what it means for f to be continuous at a! Therefore, we can conclude that if f is defined on an open interval *containing* a, then

$$f \text{ is continuous at } a \iff \lim_{x \to a} f(x) = f(a).$$

We turned to an example, showing that $\lim\limits_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$ using the principle just noted. Next, we tried computing the limit $\lim\limits_{x \to 0} \frac{|x|}{x}$, but it turns out that it does not exist!

This led into the notion of *one-sided limits*, including limits to ∞ or $-\infty$. These definitions can be thought of as adding restrictions to the x-values in the definition of a limit (e.g., $x < a$ or $x > a$), but we also gave a simpler, equivalent definition that does not require a "δ." From the definition of a one-sided limit, it is clear that if a is a real number in an open interval, where f is defined at each point in the interval except possibly at a, then

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^+} f(x) = L \text{ and } \lim_{x \to a^-} f(x) = L.$$

In particular, for a limit to exist, both two-sided limits must be equal!

Next, we showed that $\lim\limits_{x^2+5x+3} 3x^2 - 7 = \frac{1}{3}$ using the definition of a limit; this was a bit tedious! Then we stated the **Main Limit Theorem** for limits of functions approaching a real number, analogous to the Main Limit Theorem for sequences. Then we applied this theorem to the aforementioned example, quickly verifying the limit.

Lecture 14: Tuesday, March 7. Given a function $f : D \to \mathbb{R}$, we reviewed the difference between what it means for f to be continuous on D (or continuous at $a \in D$), versus f uniformly continuous on D. The difference is that we can find a "uniform" $\delta > 0$ for each $\varepsilon > 0$.

We argued, using work from last time, that the function $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$. Recall that we already showed that this function is uniformly continuous on $[1, \infty)$. Moreover, every continuous function on a closed, bounded interval is continuous on that interval, so f is uniformly continuous on $[0, 2]$. We constructed a "δ" using the two "δs" coming from each of these statements!

Given a sequence of functions $\{f_n\}$, each with domain D, we we recalled what it means for the sequence of functions to *converge pointwise*, and to *converge uniformly* to a function f with domain D. For the latter, we can find a "uniform" N for each $\varepsilon > 0$ that "works" for all points in the domain D.

Consider the sequence of functions $\{f_n\}_{n=1}^{\infty}$, where $f_n(x) = x^n$. We showed that on the domain $D = [0, 1]$, $\{f_n\}$ converges pointwise to the function

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases},$$

but that it does **not** converge uniformly to f. On the other hand, on the domain $D = [0, \frac{1}{2}]$, we showed that $\{f_n\}$ does converge uniformly to f!

With this example in mind, we stated a **theorem**: If $\{f_n\}$ is a sequence of continuous functions on a domain D, and $\{f_n\}$ converges uniformly to a function f on D, then f must be continuous on D.

Finally, we considered the sequence of functions $\{f_n\}_{n=1}^{\infty}$, where $f_n(x) = \frac{1}{1+nx}$. We first showed that each f_n is continuous on $[0, \infty)$. Then, we calculated that $\{f_n\}$ converges pointwise to

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}.$$

Since f is not continuous at $x = 0$, we can use the theorem above to conclude that $\{f_n\}$ does not converge uniformly to f on $[0, \infty)$.

The theorem we invoked gives a criterion for a sequence of functions to *not* converge uniformly to its pointwise limit. For next time, remember to read about a theorem tat the end of the section that guarantees uniform continuity.

Lecture 13: Thursday, March 2. First, we recalled what it means for a function to be **uniformly continuous** on a domain. We sketched $f(x) = \frac{1}{x}$ and $g(x) = \sqrt{x}$, and noticed the differences between the two.

We proved that $f(x) = \frac{1}{x}$ is *not* uniformly continuous on the interval $(0, 1]$, but that it is uniformly continuous on the interval $[2, 3]$.

We next stated a **theorem** that a continuous function on a closed, bounded interval is also uniformly continuous on this interval.

Next, we proved that $g(x) = \sqrt{x}$ is uniformly continuous on a non-bounded interval, $[1, \infty)$.

From here, we stated and proved the following **theorem**: If f is uniformly continuous on a domain D and $\{x_n\}$ is a Cauchy sequence in D, then the sequence $\{f(x_n)\}$ is also a Cauchy sequence.

We defined the **closure** $\overline{I}$ of an interval I, which is another interval with its finite endpoints included. We were then able to state the following **theorem**: Suppose that a function is continuous on a (not necessarily closed) interval I. Then f has a continuous extension to $\overline{I}$ if and only if f is continuous on I. We gave examples where this extension does not exists (e.g., $f(x) = \frac{1}{x}$ on $(0, 1]$), and where it does ($g(x) = \frac{x^2+x}{x}$ on $(0, 1]$).

From here, we turned to discuss **sequences of functions**. We defined what if means for a sequence of functions $\{f_n\}$, where $f_n : D \to \mathbb{R}$, to **converge pointwise** to a function $f : D \to \mathbb{R}$. Then we defined what it means for $\{f_n\}$ to **converge uniformly** to f.

For next time, practice proving that a function is (or is not) uniformly continuous on a domain. Then read about general limits in the next section.

Lecture 12: Tuesday, February 28. We started class by restating the theorem characterizing continuity of a function at a point in terms of sequences convergence.

Next, we turned to investigating the domain of the sum, product, quotient, and composition of two functions with given domains. Then, we stated a **theorem** that says that if f and g are functions and $x = a$ is a point in the domain of both, then if both f and g are continuous at a, then so are the functions cf, $f + g$, and fg, where $c \in \mathbb{R}$ is any constant. Moreover, if $g(a) \neq 0$, $\frac{f}{g}$ is also continuous at a.

We proved the first two parts of this theorem using the $\delta - \varepsilon$ definition of a limit, and then using the firs theorem we stated today. The second was was easier!

Next, we stated a **theorem** that if g is continuous at a and f is continuous at $g(a)$, then the composition $f \circ g$ is continuous at a. We motivated the hypotheses, and did some examples illustrating this, and the previous, theorem.

We recalled what it means for a function to be **bounded above**, **bounded below**, and **bounded** on a subset of its domain. We did several examples, computing whether the function $f(x) = \frac{1}{x}$ is bounded above on different subsets of its domain $\mathbb{R} \setminus \{0\}$.

Next, we stated the following **theorem**: If f is continuous on a closed, bounded interval I, then f is bounded on I, and attains its maximum and minimum values on I. We discovered that without each hypothesis, we do not necessarily get the conclusions by looking back at our example $f(x) = \frac{1}{x}$.

From here, we stated the **Intermediate Value Theorem (IVT)**: Suppose that f is a function defined on an interval containing $x = a$ and $x = b$, where $a < b$. If y is any value between $f(a)$ and $f(b)$, then there is some c in the interval $[a, b]$ such that $f(c) = y$. We sketched two graphs, the first motivating why this holds for f continuous, and the second showing that the conclusion need not hold if f is discontinuous at a point between a and b.

Finally, we used the IVT and a theorem from earlier today to show that if f is continuous on a closed, bounded interval I, then $f(I)$ is either closed and bounded, or consists of one point. We did a couple examples to illustrate this.

Lecture 11: Tuesday, February 21. We started class by showing one method for proving the following proof-writing assignment: If $\{a_n\}$ and $\{b_n\}$ are sequences, $\{a_n\}$ converges to zero and $\{b_n\}$ is bounded, then $\{a_nb_n\}$ converges to zero.

Next, we recalled the definition of **continuity** of a function at a point in its domain. We played the "$\varepsilon - \delta$" game with two functions with domain $\mathbb{R}$. Starting with $f(x) = x + 10$, we found that $\delta = \varepsilon$ works for all real numbers a, and for $g(x) = 3x$, we found that $\delta = \varepsilon/3$ works for all values of a.

Next, we considered the following piecewise function with domain $\mathbb{R}$:

$$h(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 + 1 & \text{if } x \geq 0 \end{cases}$$

We proved that $h(x)$ is *not* continuous at $x = 0$ by fixing $\varepsilon = \frac{1}{2}$ (not that even $\varepsilon = 1$ would work), and showing that no matter how small $|x|$ is forced to be, we can still such an x value for which $|h(x) - 1| > \varepsilon = \frac{1}{2}$.

We turned to showing that more complicated functions are continuous at certain points. We proved that $f(x) = \frac{1}{1+x}$ is continuous at $x = 1$; once we were given $\varepsilon > 0$, our choice of δ was $\min\{1, 2\varepsilon\}$. We also proved that $g(x) = \sqrt{x}$ is continuous at $x = 4$ using $\delta = \min\{1, (\sqrt{3} + 2)\varepsilon\}$. Then we showed that $g(x) = \sqrt{x}$ is continuous, meaning that it is continuous at every point of its natural domain $[0, \infty)$! We needed to be careful with our choice of δ; it ended up relying on a **and** on ε:

$$\delta = \begin{cases} \min\{1, (\sqrt{a-1} + \sqrt{a})\varepsilon\} & \text{if } a > 1 \\ \min\{1, \sqrt{a}\varepsilon\} & \text{if } 0 \leq a < 1 \end{cases}$$

Finally, we stated a **theorem** that a function $f : D \to \mathbb{R}$ is continuous at a point $a \in D$ if and only if for any sequence $\{x_n\}$ converging to a, the sequence $\{f(x_n)\}$ converges to $f(a)$.

Using this theorem and the Main Limit Theorem, we proved that $f(x) = x^r$ is continuous on its natural domain for every rational number r. Note that if $r = \frac{p}{q}$ is written in lowest terms with p, q integers, then the domain of f is $\mathbb{R}$ if q is odd, and is $[0, \infty)$ if q is even.

Lecture 10: Thursday, February 16. We started off by recalling the definition of a limit of a sequence, and also introduced the "$\varepsilon - N$ game" formulation of this definition. This is a two-player game between P1 and P2 to check whether $\lim\limits_{n\to\infty} a_n = L$.

1. P1 declares a distance $\varepsilon > 0$.
2. After doing side work, P2 responds with an integer N such that $|L - a_n| < \varepsilon$ whenever $n > N$.

If P2 cannot respond (i.e., if such an N does not exist for the declared distance ε) then P1 wins. Otherwise, P2 wins. If P2 always wins no matter what distance P1 declares, then $\lim\limits_{n\to\infty} a_n = L$. For practice, we played the $\varepsilon - N$ game to show that $\lim\limits_{n\to\infty} \frac{n^2+1}{n^2} = 1$.

We moved on to discussing function notation. Recall that

$$f : D \to \mathbb{R} \quad \text{and} \quad D \xrightarrow{f} \mathbb{R}$$

both mean that f is a function whose domain is a subset $D \subseteq \mathbb{R}$ and such that $f(x) \in \mathbb{R}$ for all $x \in D$. After discussing different classes of functions (e.g., polynomials, rational functions, trig functions and their inverses) and their domains, we then gave the following definition of continuity at a point.

Definition: If $f : D \to \mathbb{R}$ and $a \in D$, then f is **continuous** at a if given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta$.

We graph a graphical interpretation of this definition, and showed how it is consistent with our intuition of what "continuous" means in terms of graphs of functions. In analogy with the previous game, we introduced the "$\varepsilon - \delta$" game at a point $a \in D$. This is a two-player game between P1 and P2 to check whether $f : D \to \mathbb{R}$ is continuous at a point $a \in D$.

1. P1 declares a distance $\varepsilon > 0$.
2. After doing side work, P2 responds with a distance $\delta > 0$ such that $|f(x)-f(a)| < \varepsilon$ whenever $|x - a| < \delta$.

If P2 can respond, then P2 wins. If P2 can *always* win no matter what distance P1 declares, then f is continuous at $a \in D$.

We stressed that the outcome of this game depends on the value of a. That is, for some a, P1 can always win, but for other values of a, P1 may sometimes lose. We then played the $\varepsilon - \delta$ game for various a for the function $f = x^2$ with domain $D = \mathbb{R}$. After this, we then played it for a general $a \in D$, and we saw that if P1 declares an arbitrary distance $\varepsilon > 0$, then P2 can always respond with

$$\delta = \min\left\{1, \frac{\varepsilon}{2a+1}\right\}.$$

We did something similar with the function $f(x) = 1/x$ with domain $D = \mathbb{R} \setminus \{0\}$ to show that when $a = \pi$, then if P1 declares an arbitrary distance $\varepsilon > 0$, then P2 can always respond with

$$\delta = \min\left\{\pi - 3, \frac{\varepsilon}{3\pi}\right\}.$$

Both of these examples illustrated the following **important point**: The response δ often depends on both ε *and* the point $a \in D$ you are checking continuity at!

Lecture 9: Tuesday, February 14. We started class by clarifying the definition of extension of the Monotone Convergence Theorem (MCT) to limits in the extended real numbers. A limit can have a *possibly infinite* limit, but we only say that a sequence *converges* if its limit is finite.

We stated a theorem that says that if a sequence has a (possibly infinite) limit, then every subsequence of this sequence has the same limit. From here, we stated the **Bolzano-Weierstrass Theorem**, which says that every bounded sequence has a convergent subsequence. Due to the bounded assumption, the limit of a convergent subsequence in this theorem must be finite.

We gave several examples of sequences, determined whether they are bounded, and if so, found one (or more) convergent subsequences. We know at least one exists by the Bolzano-Weierstrass Theorem!

Next, we gave some important ideas from the proof of the Bolzano-Weierstrass Theorem.

After this, we defined what it means for a sequence to be **Cauchy**, and then proved that the sequence $\{\frac{1}{2^n}\}$ is Cauchy.

The notion of Cauchy sequences is important due to the following **theorem**: a sequences is Cauchy if and only if it converges. We proved the direction that says that a convergent sequence must be Cauchy.

Associated to any sequence $\{a_n\}$, there are two intrinsically-defined sequences, one of which is non-increasing (which we called $\{s_n\}$), and one of which is non-decreasing (which we called $\{i_n\}$).

If the original sequence is bounded, then both are also bounded, so both converge by the MCT. The limits of these sequences are the **limit superior**, $\limsup a_n$, and the **limit inferior**, $\liminf a_n$, of the original sequence. We noticed that if the original sequence is bounded, then so are these limits. We computed some examples of finding these values.

A **subsequential limit** of a sequence is the limit of a subsequence of the original sequence. We did some examples of finding some subsequential limits of sequences. They we stated the fact that every subsequential limit is bounded between the limit superior and the limit inferior of the sequence. Moreover, we know that these bounds are themselves subsequential limits!

We stated and proved a theorem that a sequence has a certain limit if and only if its limit inferior and limit interior equal that limit value.

Lecture 8: Thursday, February 9. We began class by re-stating the Monotone Convergence Theorem (MCT). We went through a full example, showing that the sequence recursively defined by

$$a_1 = 1, \text{ and } a_{n+1} = \frac{a_n}{1+2^n} \text{ for } n \geq 1$$

is decreasing and bounded, so it must converge. We did so by proving by induction on n that $a_n > a_{n+1} > 0$. We noted that the MCT does not tell us what value it converges to, but since we showed that $0 < a_n < 1$ for all $n \in \mathbb{N}$, we argued that the limit is in the interval $[0, 1]$.

Next, we defined what it means for a **limit to be infinite**: Given a sequence $\{a_n\}$, we say that $\lim_{n\to\infty} a_n = \infty$ if the following holds: Given any real number M, there is some real number N for which

$$a_n > M \quad \text{whenever} \quad n > N.$$

Alternately, we say that $\lim_{n\to\infty} a_n = -\infty$ if given any real number M, there is some real number N for which

$$a_n < M \quad \text{whenever} \quad n > N.$$

Here, just like we did to define suprema and infinima, we are extending the notion of a limit to the extended real number system. In fact, the MCT has an extension in this setting: If we include infinite limits, we can say that any (not-necessarily bounded) monotone sequence has a limit!

Note that we only say that a sequence **converges** if it has a *finite* limit!

We sketched graphs, and saw that the analogue of our infinite bars of width 2ε about a finite limit is the "half-plane" above the line $y = M$.

We used this definition to prove that for any real power $p > 0$, $\lim_{p\to\infty} n^p = \infty$. (Given M, our value of N was $N = M^{1/p} = \sqrt[p]{M}$.)

Next, we stated some properties involving infinite limits, and proved one: If $\{a_n\}$ is a positive sequence, then if its limits is ∞, then the limit of the sequence $\{\frac{1}{a_n}\}$ is zero. (This is actually an "if and only if" statement.) We noticed that in proving this, we needed to use both the definition of a finite limit, and an infinite limit.

Finally, we defined a **subsequence**, and after considering some examples, we state the **Bolzano-Weierstrass Theorem**: Every bounded sequence has a convergent subsequence.

For next time, check out the definition of a **Cauchy sequence**.

Lecture 7: Tuesday, February 8. Today, we again reviewed the definition of a limit. We proved that $\lim_{n\to\infty} \frac{n^3}{3n^3-4n} = \frac{1}{3}$, which took an extra step compared to our examples last time.

We gave an alternative definition for $\lim_{n\to\infty} a_n = L$, where L is a real number: Given any real number $\varepsilon > 0$, there only finitely many values of n for which $|a_n - L| \geq \varepsilon$.

We translated this, and the usual definition of a limit to state precisely what $\lim_{n\to\infty} a_n \neq L$ means mathematically, and we applied this in an example.

Next, we proved the **Squeeze theorem** for limits, and stated and motivated the **Main Limit Theorem**, which has several parts. We applied several of these parts to show that

$$\lim_{n\to\infty} \frac{7n^5 + 3n^4 + 2}{3n^5 - 2n^2 + 1} = \frac{7}{3}$$

without using the formal definition of a limit. (We did use that $\frac{1}{n} \to 0$ as $n \to \infty$, which we proved using this formal definition.)

Next, we defined what it means for a sequence to be **increasing**, **decreasing**, **non-increasing**, or **non-decreasing**. A **monotone** (or **monotonic**) sequence is one that is either non-increasing or non-decreasing.

We stated the **Monotone Convergence Theorem** (MCT): A bounded monotonic sequence converges.

We noted that given the sequence recursively defined by

$$a_1 = 0 \text{ and } a_{n+1} = \frac{a_n + 1}{2} \text{ for } n \geq 1,$$

it can be proved by induction on $n \in \mathbb{N}$ that $a_n \leq a_{n+1} < 1$ for every natural number n. The first inequality tells us that $\{a_n\}$ is non-decreasing, and the second tells us that it is bounded above by 1. Since it is non-decreasing, it is also bounded below by its first term $a_1 = 0$. Therefore, the sequence converges by the MCT! We noted, however, that the theorem does not tell us the value of the limit.

Lecture 6: Thursday, February 2. We started class today by recalling the definition of a limit of a sequence.

We proved, using this definition that $\lim_{n\to\infty} \frac{1}{n^2} = 0$, that $\lim_{n\to\infty} \frac{n}{3n+5} = \frac{1}{3}$, and that $\lim_{n\to\infty} \sqrt{9 + \frac{1}{n}} = 3$. We also showed that the sequence $a_n = (-1)^n$ has **no limit** using a proof by contradiction, and our definition of limit.

We defined a sequence to be **bounded above** if the set of its values is bounded above, and **bounded below** analogously. A sequence is **bounded** if it is both bounded above and below. We proved, using the definition of a limit, that a convergent sequence is bounded.

Finally, we proved that if the sequence $\{a_n\}$ converges to limit L, then the sequence whose terms are $|a_n|$ converges to limit $|L|$. This used the second part of the triangle inequality.

Lecture 5: Tuesday, January 31. We started class by coming back to suprema and infima, defining them for a function between sets, with a possible argument of a subset of the domain of the function.

We recalled the definition of the absolute value of a real number, and showed that for real numbers $x, a,$, and ε, where $\varepsilon > 0$.

$$|x - a| < \varepsilon \;\text{ if and only if } a - \varepsilon < x < a + \varepsilon.$$

In other words, x is *within* ε of a. Thus, a small value of ε ensures that x is close to a.

We proved the first part of the **triangle inequality**: For real numbers a and b, we always have:

- $|a + b| \leq |a| + |b|$
- $||a| + |b|| \leq |a - b|$

Next, we turned to sequences, giving notation, and practicing finding explicit descriptions for sequences in two examples. We noticed that every sequence can be expressed in multiple ways, including after renumbering its indices.

We gave the rough idea of what the limit of a sequence is (if it exits), and how the precise definition is (literally) more precise than our general idea of a limit. Then we stated the definition: A sequence $\{a_n\}$ **converges** to a real number L, called its **limit**, if the following holds: Given any $\varepsilon > 0$, there is some real number N for which

$$|a_n - L| < \varepsilon \;\text{ whenever }\; n > N.$$

In this case, we write $\lim\limits_{n\to\infty} a_n = L$ or say that $a_n \to L$ as $n \to \infty$.

We drew some pictures, graphing values of a sequence a_n with respect to inputs $n \in N$ that vary along the x-axis. The sequence limits to L means that for any $\varepsilon > 0$, eventually all points on this graph are in the horizontal bar of width 2ε centered at $y = L$. The value of N can be any x value such that beyond this value, all points are inside the bar. We noticed that we can always take N to be a natural number.

Finally, we proved, using the definition of a limit, that if $a_n = \frac{1}{n}$, then $\lim\limits_{n\to\infty} \frac{1}{n} = 0$.

Remember to study the definition of the limit of a sequence for next time!

Lecture 4: Thursday, January 26. We started class by recalling the definition of a Dedekind cut of the rational numbers $\mathbb{Q}$, and the fact that for every rational number, there is a canonical (intrinsically defined) Dedekind cut. We also discovered another Dedekind cut during our last class period that does not come from a rational number.

In fact, every Dedekind cut can be associated to a real number, and vice versa: there is a correspondence between them. For $r \in \mathbb{Q}$, the Dedekind cut L_r defined last time corresponds to the rational number r. The "new" Dedekind cut L from last time corresponds to the real number $\sqrt{2}$ that is not rational. We will denote by L_x the Dedekind cut associated to the real number x.

Under this correspondence, we can define the order on the real numbers (as Dedekind cuts):

$$x \leq y \iff L_x \subseteq L_y.$$

We saw that this makes sense based on the picture when x and y are rational.

Moreover, $x + y$ corresponds to the set

$$L_x + L_y = \{r + s \mid r \in L_x, s \in L_y\},$$

which is, in fact, a Dedekind cut! (You will have a chance to check this in homework.) Similarly, xy is defined via a new Dedekind cut denoted $L_x \cdot L_y$; you can check out the definition in the book.

From here, we turned to the definition of a set S being **bounded above** by a number N:

$$x \leq N \text{ for all } x \in S.$$

In this case, we call N a **upper bound** for S. If S has an upper bound, we say that it is **bounded above**, and if there is a smallest upper bound, we call it the **least upper bound** for S. We did several examples, finding the upper bound for subsets of $\mathbb{R}$, if there was one, and proving there wasn't one if not. In all cases where a set had an upper bound, it had a least upper bound. This motivated the following theorem:

Theorem. Every nonempty subset of the real numbers that is bounded above has a least upper bound.

We proved the theorem using our definition of the real numbers via Dedekind cuts of $\mathbb{Q}$.

We call the real numbers **complete** because they satisfy the property in the statement of the theorem. We noted that completeness gives the **Archimedian property** of the real numbers: for every $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ for which $x < n$.

Next, we turned to the analogue of upper bound called a **lower bound**. We figured out that the appropriate analogue of a least upper bound is a **greatest lower bound**.

We argued that every nonempty subset of the real numbers that is bounded below has a greatest lower bound by negating each element of the set and reducing to the theorem above about least upper bounds.

From here, our goal is to assign a "greatest lower bound" and "least upper bound" to *every* nonempty subset of $\mathbb{R}$, regardless of whether it is bounded below or above. For this reason, we defined the **extended real number system**, which includes two "new" numbers denoted "∞" and "$-\infty$. After some discussion, we used this new system to define the **supremum** of any subset S of $\mathbb{R}$, $\sup(S)$, an extension of the notion of a least upper bound, and the **infimum** of S, $\inf(S)$, an extension of the notion of the greatest lower bound. We did several examples, finding the supremum and infimum of sets.

Finally, for sets $S, T \subseteq \mathbb{R}$, we defined new sets called $-S$, $S + T$, and $S - T$, and then stated some relationships between supremums and infimums.

Lecture 3: Tuesday, January 24. Continuing toward building the real number system, we first described how we obtain the integers $\mathbb{Z}$ from the natural numbers $\mathbb{N}$. We stated the axioms that the set of all integers $\mathbb{Z}$ are required to satisfy, based on the two operation of addition and multiplication. From here, we noticed a defect: given an integer x, there is no integer y for which $xy = 1$; in this case y would be a *multiplicative inverse* of x. If we extend the integers to include all rational numbers, $\mathbb{Q}$, each nonzero number has a multiplicative inverse. We discussed how we can extend the operations of addition and multiplication to the rationals, and also the **order** which is determined for the natural number using Peano's axioms. We stated the axioms necessary for the set of rationals to be an ordered field, and then proved some basic facts involving inequalities of rational numbers using them.

From here, we noticed that some important numbers that come up naturally (like $\sqrt{2}$ or π)are not rational, they are **irrational**. In order to "fill in" the gaps that the rationals leave, we need to build the real number system. To start doing so, we defined a **Dedekind cut** of rationals, which is a subset of $\mathbb{Q}$ that satisfies three properties. We argued that for any rational number r,

$$L_r = \{x \in \mathbb{Q} \mid x < r\}$$

is a Dedekind cut. It was a bit trickier, but we showed that

$$L = \{r \in \mathbb{Q} \mid r \geq 0 \text{ and } r^2 < 2\} \cup \{r \in \mathbb{Q} \mid r < 0\}$$

is also a Dedekind cut, but it is not L_r for any rational r. In fact, we will match each Dedekind cut with a real number; L_r represents the rational number r, and L represents $\sqrt{2}$. Next time, we will finish describing how the set of all Dedekind cuts can be thought of as our system of real numbers.

Lecture 2: Thursday, January 19. First, we recalled the definitions of the **image** and **preimage** (or **inverse image**) of a set with respect to a function. Note that the applicable sets must be subsets of either the domain, or the set of output values, depending on which of these notions we refer to.

Next, we stated a theorem relating the preimage of a union, intersection, or complement to the union, intersection, or complement of the image. We proved the first part of the theorem. Next, we stated an analogous theorem for the *image*; we noticed that some equalities of sets were replaced with subsets, a weaker statement than our first theorem. We proved the third part of this theorem.

We finished the basics of set theory by defining the **Cartesian product** two sets, or infinitely many sets indexed by the natural numbers.

We turned to the notion of "constructing" the natural numbers. We stated Peano's axioms for the natural numbers, and stressed the fact that we should think of the "successor" of a number to be one more than it, but that these axioms can be assumed without the concept of addition. Those of us familiar with the concept of induction noticed that the final axiom appears to be an application of the Principle of Mathematical Induction. We stated this principle:

The Principle of Mathematical Induction. Suppose that $\{P_n\}$ is a statement about an integer n, where $n \geq n_0$. If

- P_{n_0} is true, and
- Whenever we assume that for *some* $n \geq n_0$, P_n is true, then P_{n+1} also holds,

then we can conclude that P_n is true for all $n \geq n_0$.

We motivated the ideal of the Principle of Mathematical Induction using the analogy of an infinite stairway or row of dominos. We then proved two statements using it. Each included checking the base case, stating the so-called "inductive hypothesis," and a proof that completes the inductive step by applying the inductive hypothesis. The first statement involved divisibility, and the goal of the second was to show that a certain recursively-defined sequence is bounded above and increasing (so that it converges!).

Lecture 1: Tuesday, January 17. We started class by familiarizing ourselves with the course website and the syllabus.

Next, we introduced the notion of a **set**, which is a collection of objects called **elements**. We gave several examples that illustrate how sets can be defined, and what notation is convenient to use in different situations. We defined a **subset** of a set, and the *empty set* (or **null set**) as the set with no elements. We gave a precise mathematical definition for the **intersection** and **union** of two, or more, sets (even infinitely many!). Two sets are called **disjoint** if their intersection is the empty set. Throughout this dicussion, we gave examples.

We defined the **complement** of one set inside another, including the case when all sets are thought of as subsets of one “universal” set. We stated a theorem relation the complement of an intersection (union, respectively) to the union (intersection, respectively) of the complements, and proved it using the basic definitions introduced thus far today.

Next, we turned to some formal definitions involving **functions** from one set to another. We defined the **image** of such a function, and the image of a subset of the domain. We defined (and described in several ways) what it means for a function to be **one-to-one** (**1-1**) or **onto**. We ended the class by defining the **inverse image** of a set under a function.