Selected Homework Solutions

Math 500: Intermediate Analysis, Spring 2017

[§1.1: #2] An element $x \in A \cap (B \cup C)$ if and only if $x \in A$ and $x \in B \cup C$. This is true if and only if $x \in A$, and x is in either B or C; in other words, either x is in both A and B, or in A and C. This holds if and only if $x \in A \cap B$ or $x \in A \cap C$, i.e., $x \in (A \cap B) \cup (A \cap C)$.

[§1.1: #4] The answer is the closed interval $[0, 1]$. *Hint*: Think about why it is enough to show that:

1. The entire closed interval $[0, 1]$ is in the specified intersection. (You might want to use the fact that $[0, 1] = (0, 1) \cup \{0, 1\}$.)
2. If either $x < 0$ or $x > 1$, then x cannot be in the intersection. (Exhibit an open interval containing $(0, 1)$ but not containing x.)

[§1.1: #5] The answer is $[0, 1]$. *Hint*: Think about why it is enough to show that:

1. Every closed interval containing $(0, 1)$ contains $[0, 1]$.
2. No proper interval that is a subset of $[0, 1]$ contains $(0, 1)$. (Think about $[0, 1] \setminus (0, 1)$.)

[§1.1: #9] We prove the statement using a series of "if and only if" statements:

$$\begin{aligned} x \in f^{-1}(E \cap F) &\iff f(x) \in E \cap F \\ &\iff f(x) \in E \text{ and } f(x) \in F \\ &\iff x \in f^{-1}(E) \text{ and } x \in f^{-1}(F) \\ &\iff x \in f^{-1}(E) \cap f^{-1}(F). \end{aligned}$$

[§1.1: #13] One example is exhibited by the function $f : \mathbb{R} \to \mathbb{R}$, where $E = [-1, 1]$ and $F = [0, 1]$. What are $f(E) \setminus f(F)$ and $f(E \setminus F)$ in this case?

[§1.2: #2] *Hint*: Proceed by induction on $n \in \mathbb{N}$. You may want to use Peano's Axiom N3 for the base case of your induction, and N4 for the inductive step. Note that you may use the result of Example 1.2.5 in your inductive step as well.

[§1.2: #8] We prove the statement by induction on $n \in \mathbb{N}$.

We start with the **base case**, $n = 1$. Since $7^1 - 2^1 = 5$, the statement is true.

Now we state the **inductive hypothesis**: For some $n \in \mathbb{N}$, assume that $7^n - 2^n$ is divisible by 5.

This means that $7^n - 2^n = 5k$ for some integer k. Therefore,

$$\begin{aligned} 7^{n+1} - 2^{n+1} &= 7^n \cdot 7 - 2^{n+1} \quad \text{by the inductive hypothesis} \\ &= (5k + 2^n)7 - 2^{n+1} \\ &= 5(7k) + 2^n \cdot 7 + 2^n \cdot 2 \\ &= 5(7k) + 2^n \cdot 5 \\ &= 5(7k + 2^n). \end{aligned}$$

We conclude that $7^{n+1} - 2^{n+1}$ is a multiple of 5. Therefore, by the Principle of Mathematical Induction, $7^n - 2^n$ is a multiple of 5 for every $n \in \mathbb{N}$.

[§**1.2: #9**] We will show by induction on $n \in \mathbb{N}$ that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

We start with the **base case**, $n = 1$. The left-hand side of the equation only has the term "1," and the right-hand side equals $\frac{1 \cdot 2}{2} = 1$, so the statement holds.

Now we state the **inductive hypothesis**: For some $n \in \mathbb{N}$, assume that $1+2+3+\cdots+n = \frac{n(n+1)}{2}$.

Then

$$\begin{aligned} 1 + 2 + 3 + \cdots + (n+1) &= (1 + 2 + 3 + \cdots + n) + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \quad \text{by the inductive hypothesis} \\ &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2}, \end{aligned}$$

and we have completed the inductive step. Therefore, by the Principle of Mathematical induction, $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for every $n \in \mathbb{N}$.

[§**1.2: #12**] By induction on $n \in \mathbb{N}$, we will prove that $0 < x_n \leq x_{n+1} < 2$.

We start with the **base case**, $n = 1$. By assumption, $0 < x_1 < 2$. Moreover, using the recursive definition of the sequence, $x_2 = \sqrt{x_1 + 2}$, and

$$0 < x_1 < 2 \implies 2 < x_1 + 2 < 4 \implies \sqrt{2} < \sqrt{x_1 + 2} < 2;$$

i.e., $\sqrt{2} < x_2 < 2$, which ensures that $0 < x_2 < 2$ as well.

Now we state the **inductive hypothesis**: For some $n \in \mathbb{N}$, assume that $0 < x_n \leq x_{n+1} < 2$.

Therefore,

$$2 < x_n + 2 \leq x_{n+1} + 2 < 4 \implies \sqrt{2} < \sqrt{x_n + 2} \leq \sqrt{x_{n+1} + 2} < 2,$$

or, in other terms, $\sqrt{2} < x_{n+1} < x_{n+2} < 2$. Since $x_{n+1} > \sqrt{2}$ certainly ensures that $x_{n+1} > 0$, we conclude that

$$0 < x_{n+1} < x_{n+2} < 2.$$

Therefore, by the Principle of Mathematical induction, $0 < x_n \leq x_{n+1} < 2$ for every $n \in \mathbb{N}$.

[§**1.2: #14**] By induction on $n \in \mathbb{N}$, we will prove that for every $n \in \mathbb{N}$, either $x_{n+1} < x_{n+2} < x_n$ or $x_{n+1} > x_{n+2} > x_n$.

We start with the **base case**, $n = 1$. Since $x_1 = 1$ is given, we compute $x_2 = \frac{1}{2}$, and then $x_3 = \frac{1}{(3/2)} = \frac{2}{3}$. Since $x_{n+1} = \frac{1}{2} < x_{n+2} = \frac{2}{3} < x_n = 1$, the statement holds.

Now we state the **inductive hypothesis**: Assume that for some $n \in \mathbb{N}$, either $x_{n+1} < x_{n+2} < x_n$ or $x_{n+1} > x_{n+2} > x_n$.

In the first case, we have that

$$x_{n+1} < x_{n+2} < x_n \implies 1 + x_{n+1} < 1 + x_{n+2} < 1 + x_n \implies \frac{1}{1 + x_{n+1}} > \frac{1}{1 + x_{n+2}} > \frac{1}{1 + x_n}.$$

This precisely says that $x_{n+2} > x_{n+3} > x_{n+1}$.

On the other hand,

$$x_{n+1} > x_{n+2} > x_n \implies 1 + x_{n+1} > 1 + x_{n+2} > 1 + x_n \implies \frac{1}{1 + x_{n+1}} < \frac{1}{1 + x_{n+2}} < \frac{1}{1 + x_n};$$

i.e., $x_{n+2} < x_{n+3} < x_{n+1}$. In either case, we have completed the inductive step. Therefore, by the Principle of Mathematical induction, for every $n \in \mathbb{N}$, x_{n+2} is between x_n and x_{n+1}.

[§**1.3: #8**] Assume that $x > 0$ and $y > 0$. Then $x \geq 0$, $y \geq 0$, $x \neq 0$, and $y \neq 0$. Applying Axiom O5 with the statements $0 \leq x$ and $0 \leq y$, we have that $0 \leq y \leq xy$; i.e., $xy \geq 0$. Since $x \neq 0$ and $y \neq 0$, we know that $xy \neq 0$. Therefore, $xy > 0$.

[§**1.3: #9**] Suppose that $x > 0$. Assume, by way of contradiction, that $x^{-1} \leq 0$. Note that $x^2 \geq 0$ by Example 1.3.8 (b), and applying Axiom O5 to these two inequalities, we have that

$$x = x^{-1} \cdot x^2 \leq 0 \cdot x^2 = 0;$$

i.e., $x \leq 0$, a contradiction. Therefore, $x^{-1} > 0$.

[§**1.3: #10**] Assume that $0 < x < y$. Then $0 \leq x \leq y$, so that $y \geq 0$ by Axiom O3. Then by problem #9, we have that $x^{-1} > 0$ and $y^{-1} > 0$.

Using Axiom O5 applied to $x \leq y$ and $0 \leq x^{-1}$, we see that $xx^{-1} \leq yx^{-1}$; i.e., $1 \leq yx^{-1}$. Applying to this inequality and $0 \leq y^{-1}$, we then obtain the inequality $y^{-1} \cdot 1 \leq y^{-1}(yx^{-1})$; i.e., $y^{-1} \leq x^{-1}$.

Now, it is enough to show that $y^{-1} \neq x^{-1}$. By way of contradition, assume that $y^{-1} = x^{-1}$. Multiplying this equation through by xy, we find that $x = (xy)y^{-1} = (xy)x^{-1} = y$, a contradiction to our assumption that $x < y$.

[§**1.4: #1**] (a) $\varnothing$; (b) $[1, \infty)$; (c) $[2, \infty)$; (d) $[1, \infty)$

[§**1.4: #2**] (a) no upper bound (supremum is ∞); (b) 1; (c) 2; (d) 1

[§**1.4: #3**] Suppose that S is a subset of $\mathbb{R}$ that is bounded above. By the completeness axiom of the real numbers, S has a least upper bound $\ell \in \mathbb{R}$; i.e., $s \leq \ell$ for all $s \in S$, and ℓ is the smallest real number with this property. Take any real number $N \geq \ell$. Then for any $s \in S$, $s \leq \ell \leq N$, so that $s \leq N$, which means that N is also an upper bound for S. Since no upper bounds are smaller than ℓ, we conclude that all upper bounds for S are exactly the real numbers in the interval $[\ell, \infty)$.

[§**1.4: #10**] This is our first proof-writing assignment.

[§**1.5: #2**] (a) The supremum is 8 (there is no maximum) and the infimum is -2 (which is the minimum); (b) assuming n ranges through the natural numbers, the supremum is $3/2$ (which is the maximum) and the infimum is zero (although there is no minimum); (a) the supremum is $\sqrt{5}$ (and there is no maximum) and the infimum is $-\infty$ (and there is no minimum).

[§**1.5: #7**] Suppose that $A \subseteq B$. We will show that $\sup A \leq \sup B$, and leave the second statement to you. First suppose that A is bounded above. Then by the completeness axiom of the real numbers, A has a least upper bound ℓ, and $\ell = \sup A$.

We now proceed to show that B cannot have a least upper bound less than ℓ (so its least upper bound is some real number greater than ℓ, or ∞). Suppose by way of contradiction that $\sup B = N < \ell$. Since ℓ is a least upper bound for A, we k now that N cannot be an upper bound for A, so there is some element $a \in A$ for which $a > N$. Now, since $A \subseteq B$, we have that $a \in B$, so N is not an upper bound for B, a contradiction!

[§**1.5: #9**] (a) The supremum (and maximum) is 1, and the infimum (and minimum) is zero; (b) the supremum is ∞ (and there is no maximum, and the infimum (and minimum) is 3; (c) the supremum is 1 (although there is no maximum), and the infimum (and minimum) is zero.

[§**2.1: #1**]

(a) We know $|x-5|<1$ if and only if $-1<x-5<1$, which holds if and only if $4<x<6$.

(b) We apply the triangle inequality to find that

$$|x-y| = |(x-3)+(3-y)| \leq |x-3|+|3-y| = |x-3|+|y-3| < \frac{1}{2}+\frac{1}{2} = 1,$$

and conclude that $|x-y|<1$.

(c) Try using the same idea as (b).

[§**2.1: #2**] Suppose, by way of contradiction, that there is some integer x for which $|x-1|<\frac{1}{2}$ and $|x-2|<\frac{1}{2}$. The first inequality tells us that $-\frac{1}{2}<x-1<\frac{1}{2}$, so that $\frac{1}{2}<x<\frac{3}{2}$; likewise, by the second, $-\frac{1}{2}<x-2<\frac{1}{2}$, so that $\frac{3}{2}<x<\frac{5}{2}$. In particular, we conclude that $x<\frac{3}{2}$ and $\frac{3}{2}<x$, which is a contradiction.

[§**2.1: #3**]

(a) $1,3,5,\ldots,2n-1,\ldots$

(b) $1,-\frac{1}{2},\frac{1}{4},\ldots,(-\frac{1}{2})^{n-1},\ldots$

(c) $1,\frac{1}{2},\frac{1}{6},\ldots,\frac{1}{n!},\ldots$

[§**2.1: #4**] We claim that $\lim\limits_{n\to\infty}\frac{1}{n^2}=0$. Fix any $\varepsilon>0$. We need to find N for which

$$\left|\frac{1}{n^2}\right| < \varepsilon \quad \text{whenever} \quad n>N.$$

Let $N=\frac{1}{\sqrt{\varepsilon}}$. Then if $n>N$, we have that

$$\left|\frac{1}{n^2}\right| = \frac{1}{n^2} < \frac{1}{N^2} = \frac{1}{(1/\sqrt{\varepsilon})^2} = \frac{1}{1/\varepsilon} = \varepsilon;$$

in particular, $\left|\frac{1}{n^2}\right|<\varepsilon$, and we are done.

[§**2.1: #5**] The limit is $\frac{2}{3}$. We first notice that for any $n\in\mathbb{N}$,

$$\left|\frac{2n-1}{3n+1}-\frac{2}{3}\right| = \left|\frac{6n-3}{9n+3}-\frac{6n+2}{9n+3}\right| = \left|\frac{6n-3-(6n+2)}{9n+3}\right| = \left|\frac{-5}{9n+3}\right| = \frac{5}{9n+3} < \frac{5}{9n}.$$

Now, fix any $\varepsilon>0$. If $N=\frac{5}{9\varepsilon}$ and $n>N$, then we can conclude that

$$\left|\frac{2n-1}{3n+1}-\frac{2}{3}\right| < \frac{5}{9n} < \frac{5}{9N} = \frac{5}{9\left(\frac{5}{9\varepsilon}\right)} = \varepsilon.$$

[§**2.1: #8**] *Hint*: Multiply the absolute value quantity by $\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = 1$, and bound your simplified expression by a fraction involving one term in the denominator. One value of N that works is $\frac{1}{\varepsilon^2}$.

[§**2.1: #11**] Suppose that $\lim_{n\to\infty} a_n = 0$, and take k any constant. We want to show that $\lim_{n\to\infty} ka_n = 0$. Fix any $\varepsilon > 0$. Then we need to find a real number N for which

$$|ka_n| < \varepsilon \quad \text{whenever} \quad n > N.$$

First, consider the real number $\frac{\varepsilon}{|k|}$ is positive. Notice that since $\lim_{n\to\infty} a_n = 0$, there is some real number M for which

$$|a_n| < \frac{\varepsilon}{|k|} \quad \text{whenever} \quad n > M.$$

Now, let $N = |k| \cdot M$. Then if $n > M$, we have that

$$|ka_n| = |k| \cdot |a_n| < |k| \cdot \frac{\varepsilon}{|k|} = \varepsilon,$$

and we are done.

[§**2.2: #2**] *Hint*: Notice that $\frac{n}{n^2+2} < \frac{1}{n}$ for all $n \in \mathbb{N}$.

[§**2.2: #5**] *Hint*: Try to prove that the limit is $\frac{1}{2}$.

[§**2.2: #6**] *Hint*: First find the limit of $1 + \frac{1}{n}$ as $n \to \infty$. Then think about the following: $a_n \to L_2$ and $b_n \to L_2$, then what value does $a_n b_n$ converge to?

[§**2.2: #9**] The sequence has no limit; its values range among the set $\{-1, -\frac{1}{2}, \frac{1}{2}, 1\}$, and each value is achieved infinitely many times. We can show that the sequence has no limit by way of contradiction: Assume that $\lim_{n\to\infty} \cos(n\pi/3) = L$ for some real number L. Then for $\varepsilon = \frac{1}{2}$, there must be some N for which

$$|\cos(n\pi/3) - L| < \varepsilon = \frac{1}{2} \quad \text{whenever} \quad n > N.$$

Since there exist natural numbers $n_1, n_2 > N$ for which $\cos(n_1\pi/3) = 1$ and $\cos(n_2\pi/3) = -1$, this means that

$$|1 - L| < \frac{1}{2} \quad \text{and} \quad |-1 - L| = |1 + L| < \frac{1}{2}.$$

Therefore, by the triangle inequality,

$$2 = |(1 - L) + (1 + L)| \le |1 - L| + |1 + L| \le \frac{1}{2} + \frac{1}{2} = 1;$$

i.e., $2 < 1$, a contradiction.

The sequence has no limit; its values range among the set $\{-1, -\frac{1}{2}, \frac{1}{2}, 1\}$, and each value is achieved infinitely many times.

[§**2.3: #4**] *Hint*: Try bounding $|\sin(n)|$ by 1 in your calculation. One valid value for N is $N = \frac{1}{\varepsilon}$.

[§**2.4: #1**] (a) non-decreasing; (b) non-increasing and bounded; (c) bounded; (d) non-increasing and bounded; (e) non-decreasing and bounded

[§**2.4: #2**] After checking a few values, we might guess that the limit is $\sqrt{3}$. To show that the sequence converges, we can use induction (as explained in Example 1.2.11) to show that the sequence satisfies the hypotheses of the Monotone Convergence Theorem: it is non-decreasing (in fact, it is increasing) and bounded by 1 from below and 2 (or even $\sqrt{3}$) from above.

[§**2.4: #3**] *Hint*: Try using induction on n that the sequence satisfies the hypotheses of the Monotone Convergence Theorem.

[§**2.4: #5**] *Hint*: Try applying the Monotone Convergence Theorem. Note what happens when the limit is not finite!

[§**2.4: #8**] Fix and real number M. We must find N for which $\frac{n^5+3n^3+2}{n^4-n+1} > M$ whenever $n > N$.
First notice that since $n-1 \geq 0$, $-(n-1) \leq 0$, and

$$\frac{n^5+3n^3+2}{n^4-n+1} = \frac{n^5+3n^3+2}{n^4-(n+1)} \geq \frac{n^5+3n^3+2}{n^4} \geq \frac{n^5}{n^4} = n.$$

Let $N = M$. Then if $n > N = M$, we have that

$$\frac{n^5+3n^3+2}{n^4-n+1} \geq n > N = M,$$

and we are done.

[§**2.4: #9**] *Hint*: First show by induction on n that $2^n > n^2$ for n large enough.

[§**2.4: #11**] *Hint*: Let your "new" value for M be the negative of the given one.

[§**2.5: #5**]

(a) Since any subsequence has infinitely many terms either of the form 2^n or $(-2)^n$, no subsequence cannot have a finite limit; i.e., no subsequence can converge.

(b) Notice that $\frac{5+(-1)^n n}{2+3n} \leq \frac{5+n}{2+3n} < \frac{5n+n}{3n} = \frac{5n}{3n} = \frac{5}{3}$, this sequence is bounded, so must have a convergent subsequence by the Bolzano-Weierstrass theorem.

(c) This sequence alternates between $2^{-1} = \frac{1}{2}$ and 2, so it has a convergent subsequence consisting of its odd-indexed, or even-indexed terms. Alternatively, we can notice that since $-1 \leq (-1)^n \leq 1$, we have that $\frac{1}{2} = 2^{-1} \leq 2^{(-1)^n} \leq 2^1 = 2$, so that $\frac{1}{2} \leq a_n \leq 2$. This means that the sequences is bounded, so it must have a convergent subsequence by the Bolzano-Weierstrass theorem.

[§**2.5: #6**] Here, we give one possible subsequence. See if you can find another!

(a) The subsequence consisting of its odd-indexed terms

(b) The subsequence whose indices are multiples of four

(c) The subsequence whose indices are powers of two

[§**2.5: #7**] To check your answers, we include the final conclusions.

(a) Two values

(b) Four values

(c) Infinitely many values

[§**2.5: #8**] It must have one, since it is bounded: $-1 \leq \sin(n) \leq 1$ by the Bolzano-Weierstrass Theorem. However, defining such a subsequence is not straightforward! Think about whether you can construct one.

[§**2.5: #9**] *Hint*: Try proving by induction that for any $k \geq 1$, $|a_{n+k} - a_n| \leq 2^{-(n+k)}$. Then notice that for arbitrary integers n and m, if $m \geq n$, then $m = n + k$ for some $k \geq 0$.

[§**2.6: #1**] (a) $\limsup a_n = 1$ and $\limsup a_n = -1$; (b) $\limsup a_n = 0$ and $\limsup a_n = 0$; (c) (a) $\limsup a_n = \frac{\sqrt{3}}{2}$ and $\limsup a_n = -\frac{\sqrt{3}}{2}$

[§**2.6: #3**] $\limsup a_n = 1$ and $\limsup a_n = 0$. Make sure to justify these answers with mathematics!

[§**2.6: #6**] Try going back to the definition of the limit superior. What is the relationship between the sets we are taking the supremum of in each side of the equality?

[§**3.1: #1**] The domain will consist of real numbers x for which $x^2 - 1 \in [0, 1]$, so that $0 \leq x^2 - 1 \leq 1$. Equivalently, $1 \leq x^2 \leq 2$, meaning that $1 \leq x \leq \sqrt{2}$ or $-\sqrt{2} \leq x \leq -1$. Thus, the domain is $[-\sqrt{2}, -1] \cup [1, \sqrt{2}]$.

[§**3.1: #3**] For every real number x, $\frac{1}{1+x^2}$ is a real number, so the natural domain is $\mathbb{R}$. Now, take any real number a. We will show that $f(x) = \frac{1}{1+x^2}$ is continuous at $x = a$.

Fix $\varepsilon > 0$. We must exhibit $\delta > 0$ for which

$$\left|\frac{1}{1+x^2} - \frac{1}{1+a^2}\right| < \varepsilon \quad \text{whenever} \quad |x - a| < \delta.$$

We first simplify the first absolute value:

$$\left|\frac{1}{1+x^2} - \frac{1}{1+a^2}\right| = \left|\frac{1+a^2}{(1+x^2)(1+a^2)} - \frac{1+x^2}{(1+x^2)(1+a^2)}\right| = \left|\frac{a^2 - x^2}{(1+x^2)(1+a^2)}\right| = \left|\frac{(a-x)(a+x)}{(1+x^2)(1+a^2)}\right|$$
$$= \frac{|x-a||a+x|}{(1+x^2)(1+a^2)}$$

If we impose the restriction that $|x - a| < 1$, we have that $a - 1 < x < a + 1$, so that

- $(a-1)^2 < x^2 < (a+1)^2$, and $1 + (a-1)^2 < 1 + x^2 < 1 + (a+1)^2$, and
- $2a - 1 < a + x < 2a + 1$, so $|x + a| < |2a + 1|$.

Therefore, we have that

$$\left|\frac{1}{1+x^2} - \frac{1}{1+a^2}\right| = \frac{|x-a||a+x|}{(1+x^2)(1+a^2)} < \frac{|x-a||a+x|}{\left(1+(a-1)^2\right)(1+a^2)} < \frac{|x-a||2a+1|}{\left(1+(a-1)^2\right)(1+a^2)}$$

If we further impose the restriction that $|x - a| < \frac{\left(1+(a-1)^2\right)(1+a^2)}{|2a+1|\varepsilon}$, then we can conclude that

$$\left|\frac{1}{1+x^2} - \frac{1}{1+a^2}\right| < \frac{|x-a||2a+1|}{\left(1+(a-1)^2\right)(1+a^2)} < \frac{\frac{\left(1+(a-1)^2\right)(1+a^2)}{|2a+1|}|2a+1|\varepsilon}{\left(1+(a-1)^2\right)(1+a^2)} = \varepsilon.$$

Therefore, if $\delta = \min\{1, \frac{(1+(a-1)^2)(1+a^2)}{|2a+1|\varepsilon}\}$, then $\left|\frac{1}{1+x^2} - \frac{1}{1+a^2}\right| < \varepsilon$. We can conclude that $f(x) = \frac{1}{1+x^2}$ is continuous on its natural domain.

[§**3.1: #4**] *Hint*: Write out the absolute values that appear in the definition of continuity, and apply the **second** part of the triangle inequality.

[§**3.1: #8**] Please see your notes from class on Tuesday, February 21.

[§**3.1: #9**] It is not continuous on domain $\mathbb{R}$, is continuous on the domain of all non-negative real numbers, but is not continuous on the domain of all non-positive numbers. To show that it is not continuous on the appropriate domains, try using $\varepsilon = 2$ or smaller.

[§**3.2: #1**] The graph of this function is an upward-facing parabola, with x-intercepts 0 and 2. The minimum value is achieved at the bottom of the parabola, when $x = 1$: $f(1) = -1$. The function has no maximum value – its values increase as $x \to 3$.

[§**3.2: #4**] For the first parts, consider $f(x) = 3$ on $\mathbb{R}$, or $g(x) = \frac{1}{x}$ on $[1, \infty)$. For the second, consider $g(x) = \frac{1}{x}$ on the bounded interval $(0, 1)$. Make sure to find some examples of your own as well!

[§**3.2: #7**] Try constructing a piecewise function that is discontinuous at a point in $[0, 1]$, and satisfies the requirement. For example, what value(s) between $f(0)$ and $f(1)$ does the following function not take on?

$$f(x) = \begin{cases} -1 & \text{if } 0 \le x \le \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < x \le 1 \end{cases}$$

[§**3.2: #9**] We will apply the IVT to the function $g(x) = f(x) - x$, with $a = 0$ and $b = 1$. Notice that $g(0) = f(0) - 0 = f(0)$ and $g(1) = f(1) - 1$. Since the values of f are in the interval $[0, 1]$, we can conclude that

$$g(0) = f(0) \quad \text{and} \quad g(1) = f(1) - 1 \le 1 - 1 = 0.$$

Therefore, $y = 0$ is between $g(0)$ and $g(1)$; i.e., $g(1) \le 0 \le g(0)$. By the IVT, there is some value $c \in [0, 1]$ for which $g(c) = y = 0$. This means $f(c) - c = 0$, or $f(c) = c$.

[§**3.2: #11**] Let f be a polynomial of odd degree. Note that either

$$\lim_{n\to\infty} f(n) = \infty \quad \text{and} \quad \lim_{n\to\infty} f(-n) = -\infty$$

or

$$\lim_{n\to\infty} f(n) = -\infty \quad \text{and} \quad \lim_{n\to\infty} f(-n) = \infty$$

(Think about the graph of f!)

In the first case, we know that for some N, if $n > N$, then $f(N) > 0$, and for some N', if $n > N'$, then $f(-N') < 0$. By the IVT, since $f(N') < 0 < f(N)$, there is some $-N' < c < N$ for which $f(c) = 0$.

What happens in the second case?

[§**3.3: #1**] Yes, it is uniformly continuous on $(0, 1)$. Fix $\varepsilon > 0$ and $0 < x, a < 1$. Then

$$|f(x) - f(a)| = |x^2 - a^2| = |x - a| \cdot |x + a| = |x - a| \cdot (x + a) < 2|x - a|.$$

Then if $|x - a| < \frac{\varepsilon}{2}$, $|f(x) - f(a)| = 2|x - a| < 2 \cdot \left(\frac{\varepsilon}{2}\right) = \varepsilon$.

[§**3.3: #2**] No, it is not uniformly continuous on $(0,1)$. Fix $\varepsilon = 1$. For any $1 > \delta > 0$, fix $0 < a < \delta$, and let $x = \frac{a}{2}$, so that $0 < x < a < \delta$. Then $|x-a| < \delta$, and

$$\left|\frac{1}{x^2} - \frac{1}{a^2}\right| = \frac{1}{x^2} - \frac{1}{a^2} = \frac{1}{(a/2)^2} - \frac{1}{a^2} = \frac{4}{a^2} - \frac{1}{a^2} = \frac{3}{a^2} > \frac{3}{1^2} = 3 > 1 = \varepsilon.$$

[§**3.3: #3**] No, it is not uniformly continuous on $(0,\infty)$. Fix $\varepsilon = 1$. For any $\delta > 0$, fix $x > \frac{2}{\delta}$. If $a = x + \frac{\delta}{2}$, then $|x-a| = \frac{\delta}{2} < \delta$. However,

$$|x^2 - a^2| = |x-a| \cdot |x+a| = \frac{\delta}{2} \cdot |x+a| = \frac{\delta}{2} \cdot (x+a) > \frac{\delta}{2} \cdot x > \frac{\delta}{2} \cdot \frac{2}{\delta} = 1;$$

i.e, $|x^2 - a^2| > 1 = \varepsilon$.

[§**3.3: #4**] Fix any $\varepsilon > 0$ and $x, a > 0$. Let $\delta = \varepsilon$ and suppose that $|x-a| < \delta$. Since $x+1, a+1 > 0$,

$$\left|\frac{x}{x+1} - \frac{a}{a+1}\right| = \left|\frac{x(a+1) - a(x+1)}{(x+1)(a+1)}\right| = \frac{|x-a|}{(x+1)(a+1)} < \frac{|x-a|}{1 \cdot 1} < \delta = \varepsilon.$$

[§**3.3: #5**] We proved in class that $\sqrt{x}$ is continuous on $[1,\infty)$. Since $\sqrt{x}$ is continuous on the closed, bounded interval $[0,2]$, it is uniformly continuous on this interval by a theorem in the section. Now, given $\varepsilon > 0$, we can take δ to be the minimum of the two values that ensure uniform continuity on each of these intervals.

[§**3.4: #1**] Take any closed interval $[a,b]$. To show that $\{\frac{x}{n}\}$ converges uniformly to 0 on this interval, fix $\varepsilon > 0$. We need to find N for which

$$|f_n(x) - 0| = |f_n(x)| < \varepsilon \quad \text{whenever} \quad n > N \text{ and } a \le x \le b.$$

Now,

$$|f_n(x)| = \left|\frac{x}{n}\right| = \frac{|x|}{n} < \frac{|b|}{n}$$

so if $N = \frac{|b|}{\varepsilon}$, then if $n > N$, we have that

$$|f_n(x)| < \frac{|b|}{n} < \frac{|b|}{N} = \frac{|b|}{\left(\frac{|b|}{\varepsilon}\right)} = \varepsilon.$$

On the other hand, we will now show that $\{\frac{x}{n}\}$ does not converge uniformly to 0 on $\mathbb{R}$: Fix $\varepsilon = 1$. Then for *any* $N > 0$, if we choose $x = N+1$, we have that

$$|f_n(x) - 0| = \left|\frac{x}{n}\right| = \frac{|x|}{n} = \frac{N+1}{n}.$$

Thus, if we choose $n = N+1 > N$, then

$$|f_n(x) - 0| = \frac{N+1}{n} = \frac{N+1}{N+1} = 1 = \varepsilon,$$

so it is not true that $|f_n(x) - 0| < \varepsilon$.

[§**3.4: #2**] Fix any $\varepsilon > 0$. We need to find N for which

$$|\frac{1}{x^2+n} - 0| = \frac{1}{x^2+n} < \varepsilon \text{ whenever } n > N \text{ and } x \in \mathbb{R}.$$

Notice that for any $x \in \mathbb{R}$, $x^2 \geq 0$, so that

$$\frac{1}{x^2+n} \leq \frac{1}{n}.$$

Thus, if $N = \frac{1}{\varepsilon}$, then for any $x \in \mathbb{R}$, if $n > N$,

$$|\frac{1}{x^2+n} - 0| = \frac{1}{x^2+n} \leq \frac{1}{n} < \frac{1}{N} = \frac{1}{\left(\frac{1}{\varepsilon}\right)} = \varepsilon.$$

[§**3.4: #4**] *Hint*: Apply Theorem 3.4.6.

[§**3.4: #5**] *Hint*: Use the book's hint to bound the values of the function for all values of x.

[§**4.1: #1**] The limit is 2. Since for all $x \neq 1$ (i.e., on the interval $\mathbb{R} \setminus \{1\}$), $\frac{x^2-1}{x-1} = x+1$, we have that

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1}(x+1) = 1+1 = 2,$$

Since the function $y = x+1$ is a polynomial, so is continuous on $\mathbb{R}$.

[§**4.1: #3**] The limit is 4. *Hint*: Try following the same process as #1.

[§**4.1: #6**] The limit equals $\frac{1}{2}$. Try proving two ways: (1) Using the $\delta - \varepsilon$ definition of limit, and (2) Applying the Main Limit Theorems.

[§**4.1: #8**] No, neither limit exists.

[§**4.1: #14**] Suppose that f is a function defined on an open interval (a, b). We say that $\lim\limits_{x\to b^-} f(x) = -\infty$ if

[§**4.2: #1**] First, notice that the domain of $f(x) = \frac{1}{x}$ is $(\infty, 0) \cup (0, \infty)$. We find the formula for the derivative function on each of these two open intervals, so on their union: If $a \neq 0$, then

$$f'(a) = \lim_{x\to a} \frac{\left(\frac{1}{x} - \frac{1}{a}\right)}{x-a} = \lim_{x\to a} \frac{\left(\frac{a-x}{xa}\right)}{x-a} = \lim_{x\to a} \frac{-1}{xa} = -\frac{1}{a^2},$$

where the last equality holds since for any $a \neq 0$, the function $y = \frac{-1}{xa}$ is continuous at a, so this limit equals this function's value at a. Thus, $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$ for all $x \neq 0$.

[§**4.2: #2**] Try using either definition of the derivative, and doing a lot of algebra!

[§**4.2: #11**] *Hint*: Try #12 first, to get the basic idea. For the second part, you might want to (eventually) try showing that $\lim\limits_{x\to 0} x\sin(1/x) = 0$ using the $\delta - \varepsilon$ definition of a limit.

[§**4.2: #12**] We find

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x-0} = \lim_{x\to 0^+} \frac{x^2-0}{x-0} = \lim_{x\to 0^+} \frac{x^2}{x},$$

which equals $\lim\limits_{x\to 0^+} x = 0$ since the function $y = \frac{x^2}{x}$ agrees with the function $y = x$ for all $x \neq 0$. On the other hand,

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{0 - 0}{x - 0} = \lim_{x\to 0^+} 0 = 0.$$

Since these limits equal the same, finite, number zero, we know that f is differentiable at zero and $f'(0) = 0$.

[§**4.3: #1**] *Hint*: Try applying the MVT on the intervals $(-1, 0)$, $(-1, 1)$, and $(0, 1)$.

[§**4.3: #2**] *Hint*: Try apply the MVT to the function $f(x) = \sin x$, and use the fact that $f'(x) = \cos x$ is bounded in absolute value by 1.

[§**4.3: #7**] *Hint*: Find the points where the given function has a zero derivative.

[§**4.3: #8**] *Hint*: Find the points where the given function has a zero derivative.

[§**4.4: #1**] Using Cauchy's MVT with $f(x)$ and $g(x)$ as indicated on the interval $(1, x)$ for a fixed x, we know that there exists $c > 1$ for which

$$\frac{1}{rc^r} = \frac{\left(\frac{1}{c}\right)}{rc^{r-1}} = \frac{f'(c)}{g'(c)} = \frac{\ln(x) - \ln(1)}{x^r - 1^r} = \frac{\ln x}{x^r - 1}.$$

Therefore, $\ln x = \frac{x^r - 1}{rc^r} \leq \frac{x^r - 1}{r}$, where the last statement holds because $c > 1$, so that $c^r > 1^r = 1$.

[§**4.4: #6**] The limit equals zero.

[§**4.4: #7**] The limit equals zero. *Hint*: Write $x \ln x = \frac{\ln x}{1/x}$.

[§**4.4: #11**] The limit equals one. *Hint*: First find $\lim\limits_{x\to\infty} \ln(x^{1/x}) = \lim\limits_{x\to\infty} \frac{1}{x} \ln(x)$. Then use your answer to find the limit in question.

[§**4.4: #13**] The limit equals zero *Hint*: The values agree with the function $y = \frac{\ln x}{\sqrt{x}}$ for $x > 0$.

[§**5.1: #1**] If there are four subintervals, then their length is each $x_k - x_{k-1} = \frac{1}{4}$, so the partition is:

$$P = \left\{1 < \frac{5}{4} < \frac{3}{2} < \frac{7}{4} < 2\right\}$$

Moreover, since the function is decreasing on this interval, the lower sum chooses $\bar{x}_k = x_k$, and the upper sum chooses $\bar{x}_k = x_{k-1}$. Therefore, the lower sum and upper sum are:

$$L(f, P) = \frac{4}{5} \cdot \frac{1}{4} + \frac{2}{3} \cdot \frac{1}{4} + \frac{4}{7} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} = \frac{533}{840}$$
$$U(f, P) = 1 \cdot \frac{1}{4} + \frac{4}{5} \cdot \frac{1}{4} + \frac{2}{3} \cdot \frac{1}{4} + \frac{4}{7} \cdot \frac{1}{4} = \frac{319}{420}.$$

[§**5.1: #2**] Given the partition noted, we have that $x_k = \frac{k}{n}$ and $x_k - x_{k-1} = \frac{1}{n}$. Since $f(x) = x$ is increasing, $\bar{x}_k = x_{k-1} = \frac{k-1}{n}$ for the lower sum, and $\bar{x}_k = x_k = \frac{k}{n}$ for the upper sum. In each case,

$f(\bar{x}_k) = \bar{x}_k$. Therefore,

$$L(f, P_n) = \sum_{k=1}^{n} \frac{k-1}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{k=1}^{n} (k-1) = \frac{1}{n^2} \cdot \frac{(n-1)n}{2} = \frac{n-1}{2n}, \text{ and}$$

$$U(f, P_n) = \sum_{k=1}^{n} \frac{k}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{k=1}^{n} k = \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2n}.$$

Thus,

$$\lim_{n\to\infty} (U(f, P_n) - L(f, P_n)) = \lim_{n\to\infty} \frac{2}{2n} = 0.$$

Therefore, by Theorem 5.1.8,

$$\int_0^1 x\, dx = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}.$$

[§**5.1: #4**] We will use the result from #3. Notice we can choose the partition on $[0, a]$ with n subintervals of equal length, $\frac{a}{n}$; i.e., $x_k = \frac{ak}{n}$ for $1 \le k \le n$. Since the function is increasing on this interval, $\bar{x}_k = x_k = \frac{ak}{n}$ for the upper sum, and $\bar{x}_k = x_{k-1} = \frac{a(k-1)}{n}$ for the lower sum. Now, by #3,

$$U(f, P_n) = \sum_{k=1}^{n} \left(\frac{ak}{n}\right)^2 \cdot \frac{a}{n} = \frac{a^3}{n^3} \sum_{k=1}^{n} k^2 = \frac{a^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}, \text{ and}$$

$$L(f, P_n) = \sum_{k=1}^{n} \left(\frac{a(k-1)}{n}\right)^2 \cdot \frac{a}{n} = \frac{a^3}{n^3} \sum_{k=1}^{n} (k-1)^2) = \frac{a^3}{n^3} \cdot \left(\frac{n(n+1)(2n+1)}{6} - n^2\right).$$

Therefore,

$$\lim_{n\to\infty} (U(f, P_n) - L(f, P_n)) = \lim_{n\to\infty} \left(\frac{a^3}{n^3} \cdot n^2\right) = \lim_{n\to\infty} \frac{a^3}{n} = 0.$$

Then by Theorem 5.1.8,

$$\int_0^a x^2\, dx = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} \frac{a^3(n+1)(2n+1)}{6n^2} = \frac{2a^3}{6} = \frac{a^3}{3}.$$

[§**5.1: #5**] The answer is **no**! *Hint*: Try showing that for any partition, $M_k = 1$ and $m_k = 0$ always.

[§**5.1: #8**] *Hint*: Try using the most "boring" partition of $[a, b]$ possible: $P = \{a = x_0 < x_1 = b\}$.

[§**5.2: #1**] *Hint*: Try applying Theorem 5.2.1 to each of g and h, and then applying Theorem 5.2.3(b).

[§**5.2: #4**] *Hint*: Try applying Theorem 5.2.4 to the inequalities

$$f(x) \le \sup_{[a,b]}(f) \quad \text{and} \quad \inf_{[a,b]}(f) \le f(x),$$

which hold for $x \in [a, b]$.

[§**5.2: #6**] The function $f(x) = \frac{1}{1+x^{2n}}$ is continuous on the interval $[-1, 1]$, and

$$f'(x) = -(1 + x^{2n})^{-2} \cdot (2n)x^{2n-1} = -\frac{2nx^{2n-1}}{1 + x^{2n}},$$

so the only critical point is $x = 0$. Since $f(0) = 1$, $f(-1) = \frac{1}{2}$, and $f(1) = \frac{1}{2}$, $\sup_{[-1,1]}(f) = 1$ and $\inf_{[-1,1]}(f) = \frac{1}{2}$. Then the result follows from applying Corollary 5.2.5.

[§**5.2: #11**] For example, the piecewise function $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$.

[§**5.2: #12**] *Hint*: Suppose that f achieves its maximum value at x_1, and its minimum value at x_2. Apply the Mean Value Theorem to f and $y = \frac{1}{b-1}\int_a^b f(x)\,dx$ on the interval $[x_1, x_2]$. Make sure to note that Corollary 5.2.5 is needed to apply the MVT.

[§**5.3: #2**] By the Second Fundamental Theorem of Calculus, on any closed interval $[b, c]$ for which $b > 0$, this derivative equals $\cos(1/x)$. For every $x > 0$, there exist b and c for which $b < x < c$. This means that the derivative equals $\cos(1/x)$ for all $x > 0$.

[§**5.3: #3**] If $F(x) = \int_0^x \sin(t^2)\,dt$, then we know by the Second FTC that $F'(x) = \sin(x^2)$. Since $\int_0^{2x} \sin(t^2)\,dt = F(g(x))$, where $g(x) = 2x$, by the chain rule,

$$\frac{d}{dx}\left(\int_0^{2x} \sin(t^2)\,dt\right) = F'(g(x)) \cdot g'(x) = F'(2x) \cdot 2 = 2\sin((2x)^2) = 2\sin(4x^2).$$

[§**5.3: #4**] *Hint*: Try writing $\int_{1/x}^x e^{-t^2}\,dt$ as

$$\int_0^x e^{-t^2}\,dt - \int_0^{1/x} e^{-t^2}\,dt,$$

and for the second term, use a method similar to #3; i.e., try writing it as a composition of two functions that you know the derivative of, and apply the chain rule.

[§**5.3: #5**] The problem is that f is not integrable on $[-1, 1]$ (or even $[0, 1]$ or $[-1, 0]$) since it is not bounded there.

[§**5.3: #10**] The problem is that f is not differentiable on the open interval $(-1, 1)$. To rectify this, we can notice that f **is** integrable on both the interval $(-1, 0)$ and the interval $(0, 1)$, so the intervals $[-1, 0]$ and $[0, 1]$ apply to the First FTC, and we find that

$$\int_{-1}^0 f'(x)\,dx = f(0) - f(-1) = |0| - |-1| = -1, \text{ and}$$
$$\int_0^1 f'(x)\,dx = f(1) - f(0) = |1| - |0| = 1,$$

so that we can conclude the (correct) statement, that

$$\int_{-1}^1 f'(x)\,dx = \int_{-1}^0 f'(x)\,dx + \int_0^1 f'(x)\,dx = -1 + 1 = 0.$$

[§**5.3: #6**] The answer is $\frac{1}{2}(f(b))^2 - (f(a))^2)$. *Hint*: Apply integration by parts.

[§**5.3: #11**] We have five cases, $a < c < b$, $b < a < c$, $b < c < a$, $c < a < b$, and $c < b < a$. For each, we want to show that:

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

For the first, since $a < c < b$, we know that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

But then

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx - \int_c^b f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Proceed similarly for the other cases.

[§**5.3: #12**] *Hint*: When $a < b$, apply Corollary 5.2.5. When $b < a$, write $\int_a^b f(x)\,dx$ as $-\int_b^a f(x)\,dx$, and apply the same Corollary. When $a = b$, think about what the statement says.

[§**5.4: #1**] Notice that

$$\frac{d}{dx}(\ln(x^r)) = \frac{1}{x^r} \cdot r x^{r-1} = \frac{r}{x},$$

so that $r \ln x$ and $\ln(x^r)$ are both antiderivatives of $\frac{r}{x}$. Therefore,

$$r \ln x = \ln(x^r) + C$$

for some constant C. Taking $x = 1$ in this equation, we find that

$$0 = r \cdot 0 = 0 + C,$$

so that $C = 0$, and $r \ln(x) = \ln(x^r)$ for all x; in particular, this holds for $x = a$.

[§**5.4: #2**] *Hint*: For any $a > 0$, take the derivative of $\ln(a/x)$, and apply a similar method to #1 or Theorem 5.4.2.

[§**5.4: #4**] Let $x = \exp a$, so that $a = \ln x$. If we can show that $\ln(\exp(ra)) = \ln((\exp(a))^r)$, then applying exp to both sides, we get that

$$\exp(ra) = \exp(\ln(\exp(ra))) = \exp(\ln((\exp(a))^r)) = (\exp(a))^r).$$

This holds, since

$$\ln(\exp(ra)) = ra = r \ln x = \ln(x^r) = \ln((\exp(a))^r).$$

[§**5.4: #5**] We will show the first, and leave the second to you. By the definition of a^x,

$$a^{x+y} = \exp((x+y)\ln a) = \exp(x \ln a + y \ln a) = \exp(\ln(a^x) + \ln(a^y)) = \exp(\ln(a^x a^y)) = a^x a^y.$$

[§**5.4: #6**] *Hint*: Check that $\log_a(x)$ and a^x are inverses, and then differentiate $\log_a(a^x) = x$ as in Theorem 5.4.6.

[§**5.4: #10**] We did #9 in class; follow a similar process. The final answer is that the integral converges if and only if $p < 1$.

[§**5.4: #11**] We write

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,dx &= \int_{-\infty}^{0} \frac{\sin x}{1+x^2}\,dx + \int_0^{\infty} \frac{\sin x}{1+x^2}\,dx \\ &= \lim_{a\to-\infty} \int_a^0 \frac{\sin x}{1+x^2}\,dx + \lim_{b\to\infty} \int_0^b \frac{\sin x}{1+x^2}\,dx,\end{aligned}$$

and since by Theorem 5.3.6, on any finite interval $[a, b]$,

$$\left|\int_a^b \frac{\sin x}{1+x^2}\,dx\right| \le \int_a^b \left|\frac{\sin x}{1+x^2}\right|\,dx,$$

and since $|\frac{\sin x}{1+x^2}| \le \frac{1}{1+x^2}$,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,dx &\le \lim_{a\to-\infty}\int_a^0 \frac{1}{1+x^2}\,dx + \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx \\ &= \lim_{a\to-\infty}(\arctan(a)-\arctan(0)) - \lim_{b\to\infty}(\arctan(b)-\arctan(0)) \\ &= (\pi/2-0)-(0-\pi/2)=\pi.\end{aligned}$$

[§**5.4: #13**] We write

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx \\ &= \lim_{a\to-\infty}\int_a^0 f(x)\,dx + \lim_{b\to\infty}\int_0^b f(x)\,dx.\end{aligned}$$

Since $f(x) \le g(x)$, for all $a, b \in \mathbb{R}$, $\int_a^0 f(x)\,dx \le \int_a^0 g(x)\,dx$ and $\int_0^b f(x)\,dx \le \int_0^b g(x)\,dx$. Therefore, by the Main Limit Theorem,

$$\begin{aligned}\lim_{a\to-\infty}\int_a^0 f(x)\,dx &\le \lim_{a\to-\infty}\int_a^0 g(x)\,dx, \text{ and} \\ \lim_{b\to\infty}\int_0^b f(x)\,dx &\le \lim_{b\to\infty}\int_0^b g(x)\,dx.\end{aligned}$$

From here, if a left-hand limit not exists or equals ∞, then what happens to the corresponding right-hand limit?

[§**6.1: #2**] This converges by the Comparison Test, comparing each term with $\frac{1}{2^n}$, whose series converges since it is geometric and $r = \frac{1}{2}$.

[§**6.1: #5**] In a similar method to Example 6.1.12, we can show that there exists N for which $\frac{k^2}{4^{k/2}} < 1$ whenever $k > N$. Then $\frac{k^2}{4^{k/2}} < \frac{4^{k/2}}{4^k} = \frac{1}{\sqrt{4}^k}$, and a series with the latter terms converge since it is geometric and $r = \frac{1}{4}$.

[§**6.1: #11**] *Hint*: Follow the proof of Theorem 6.1.9, but simplify it by disregarding M and K.

[§**6.2: #1**] This series diverges by the Integral Test.

[§**6.2: #4**] The series converges by the Ratio Test.

[§**6.2: #9**] *Hint*: Try graphing the functions $g(x)$ and $g(x+1)$, and thinking carefully about Riemann sums.

[§**6.2: #12**] Suppose that $a_k = b_k$ for all $k > N$. Let $s_n = a_1 + a_2 + \ldots + a_n$ denote the n^{th} partial sum for $\sum_{k=1}^{\infty} a_k$, and let $t_n = b_1 + b_2 + \ldots + b_n$ denote the n^{th} partial sum for $\sum_{k=1}^{\infty} b_k$. Then for $n > N$,

$$\begin{aligned} s_n - t_n &= (a_1 + a_2 + \cdots + a_N + a_{N+1} + \cdots + a_n) - (b_1 + b_2 + \cdots + b_N + b_{N+1} + \cdots + b_n) \\ &= (a_1 + a_2 + \cdots + a_N) - (b_1 + b_2 + \cdots + b_N) \\ &= s_N - t_N, \end{aligned}$$

and $s_N - t_N$ is a constant. Therefore, $\lim\limits_{n\to\infty}(s_n - t_n) = 0$.

If $\sum\limits_{k=1}^{\infty} a_k$ converges, then for some real number s, $\lim_{n\to\infty} s_n = s$, so that

$$0 = \lim_{n\to\infty}(s_n - t_n) = s - \lim_{n\to\infty} t_n,$$

i.e., $\lim\limits_{n\to\infty} t_n = s$, and $\sum\limits_{k=1}^{\infty} b_k$ also converges.

On the other hand, if $\sum\limits_{k=1}^{\infty} a_k$ diverges, then $\{s_n\}$ has no finite limit, so that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} t_n + (s_n - t_n) = \lim_{n\to\infty} s_n$$

also does not exist, and $\sum\limits_{k=1}^{\infty} b_k$ diverges as well.

Since we can switch the roles of a_n and b_n, we are done.

[§**6.3: #4**] Diverges since its terms do not approach zero (try dividing the numerator and denominator by 2^k).

[§**6.3: #5**] Converges by the Alternating Series Test; not absolutely convergent by comparison test (what comparison series can you use?).

[§**6.3: #6**] For example, $a_k = b_k = (-1)^k \frac{1}{\sqrt{k}}$. Try finding another pair!

[§**6.3: #7**] Suppose that $\sum\limits_{k=1}^{\infty} a_k$ converges to s.

Notice that the series in question consist of all positive terms, or negative terms, respectively. Suppose, by way of contradiction, suppose that $\sum\limits_{k=1}^{\infty} a_k^+$ converges to a real number t. In this case, $\sum\limits_{k=1}^{\infty} a_k^-$ must converge, since

$$\sum_{k=1}^{\infty} a_k^- = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{\infty} a_k^+ = s - t.$$

Suppose it converges to the real number r. However, $|a_k+| = a_k$ if $a_k > 0$ and $|a_k| = -a_k$ if $k < 0$, so that

$$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} a_k^+ - \sum_{k=1}^{\infty} a_k^- = t - r$$

would also converge, contradicting the fact that the original series converges absolutely. Therefore, we can conclude that $\sum\limits_{k=1}^{\infty} a_k^+ s$ diverges. Than a symmetric argument will show the same for $\sum\limits_{k=1}^{\infty} a_k^-$.

[§**6.3: #9**] To check your method, the answer begins:

$$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} - \frac{1}{4} + \frac{1}{15} + \cdots$$

but we need more terms! Here, we continue adding positive terms so that the partial sum is within $\frac{1}{n}$ of $\sqrt{2}$; i.e., pick m so that $\sqrt{2} < s_m < \sqrt{2} + \frac{1}{n}$, and then adding the next negative term.

[§**6.4: #1**] On the interval $[-1, 1]$, $\left|\frac{x^k}{k^2}\right| = \frac{|x|^k}{k^2} \leq \frac{1}{k^2}$ converges. Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$, we can use $M_k = \frac{1}{k^2}$ in the Weierstrass M-test to conclude that the original series converges uniformly on $[-1, 1]$; therefore, it is continuous by Theorem 6.4.2 since each functino $f_n(x) = \frac{x^k}{k^2}$ is a polynomial, so is continuous on this interval.

[§**6.4: #2**] *Hint*: Use the same idea as #1, but with $M_k = \frac{1}{2^k}$.

[§**6.4: #10**] By the Alternating Series Test, for each $x \in [0, 1]$, the series of partial sums $g_n(x) = \sum_{k=0}^{n} (-1)^{k+1} a_k x^k$ converges to the value $g(x) = \sum_{k=0}^{\infty} -1)^{k+1} a_k x^k$ since the sequence $\{a_k x^k\}$ is non-increasing (since $\{x^k\}$ is, and $a_k \geq 0$) and consists of non-negative numbers (since $a_k \geq 0$). Moreover, $|a_k x^k| - a_k x^k \leq a_k$ for all $k \geq 0$ if $x \in [0, 1]$, and since $\{a_k\}$ is non-increasing, the series converges uniformly to $g(x)$ on $[0, 1]$ by the Weierstrass M-test. Then the continuity of $g(x)$ on $[0, 1]$ follows from Theorem 6.4.2.