MATH 500 Update

Fall 2023

Week 16	December 4–8
Read	§5.2, 5.3
Homework	§5.3: 1–6, 10–12
Monday	Today we stated the fact that if f is either monotone or continuous on $[a, b]$, then is is integrable on $[a, b]$, and proved the remaining part–the case that f is continuous. Next, we stated the linearity of the integral, and proved, using old theorems on sups and infs (!) that one can "pull out" scalars. Finally, we stated the first fundamental theorem of calculus (FTC I), and showed an example of a function that is differentiable on $[a, b]$ but its derivative is not continuous on $[a, b]$.
Wednesday	Today we proved the first fundamental theorem of calculus. Next, we stated the second fundamental theorem of calculus, used it in an example, and then proved it. Thanks for a great semester!
Friday	Stop Day

Week 15	November 17–December 1
Read	§5.1, 5.2
Homework	§5.1: 1–3, 4—9; §5.2: 1
Monday	Today we recalled the definition of a partition of a closed interval, a Riemann sum of a bounded function on the interval, and the upper and lower sums of the function on the interval with respect to such a partition. We stated a theorem and proved a corollary, both comparing upper and lower sums of different partitions. Finally, we defined the upper and lower integrals, and what it means for a function to be differentiable on a lower interval.
Wednesday	We started class by reviewing the definition of a partition of a closed interval, and the upper and lower sums of a bounded function on the interval with respect to such a partition. The we proved the theorem from last time in the case that the refinement of the partition has only one extra element. We used the theorem to prove a theorem stating that a function is integrable if and only if there is a partition for which the upper and lower sums are as close as desired. Finally, we stated a theorem characterizing integrability in terms of sequences of partitions.
Friday	After our last quiz–Quiz 10–on integration, we stated two theorems from last time, and used the first to prove the second, characterizing integrability using sequences of partitions. We used the theorem to show that x^2 is integrable on the unit interval, and computed its integral. Next, we stated and proved the fact that any monotone function on a closed interval is integrable there, and stated the fact that any continuous function on a closed interval is also integrable on the interval.

Week 14	November 20–24
Read	§4.3, 5.1
Homework	§4.3: 6–10; 5.1: 1, 3, 6
Monday	Today we derived some additional consequences of the MVT, on the derivative of f and its relation to the increasing/decreasing nature of f. Next, we started Chapter 5 on Integrals. We started by defining a partition of a closed interval, and a Riemann sum of a bounded function on the interval with respect to such a partition. We used this to define the upper and lower sum of a function on an interval, with respect to a given partition.
Wednesday	Thanksgiving Break
Friday	Thanksgiving Break

Week 13	November 13–17
Read	§4.2, 4.3, 5.1
Homework	§4.2: 4, 6–8, 10; 4.3: 1–4
Monday	Today we recalled the definition of the derivative of a function at a point. We proved that $\sin x$ is differentiable at every real number, so that our theorem from last time says it is also continuous everywhere. Next, we stated and proved a theorem stating linearity property of derivatives, along with the product and quotient rule. Finally, we stated the Chain Rule, which has several important hypotheses.
Wednesday	We started class with Quiz 9 on differentiation. Next, we recalled the Chain Rule, and used it to establish the formula for the derivative of an inverse function. We used this formula to re-derive the formula for the derivative for $\arcsin x$.
Friday	Today, we defined critical points of a function on a closed interval, and proved that if the function attains a minimum or maximum at a point in that interval, it must be a critical point. Then we used this results to prove the Mean Value Theorem (MVT) relating the derivative of a function to the slope of the secant line to a function between two points in its domain. We used the MVT to find an easy proof of the familiar statement that if the derivative of a function is zero on an open interval, the function must be constant!

Week 12 November 6–10

Read §4.1, 4.2

Homework §4.1: 1–10, 15; §4.2: 1–3, 9

Monday Today, we defined what it means for a function f defined on an open interval, except possibly a point a in the interval, to have limit L as x approaches a, i.e., $\lim_{x \to a} f(x) = L$. This was very similar to our definition of limits of sequences. We went through several examples, and then defined limits from the right and from the left, which includes limits as $x \to \pm\infty$. We related the definitions, in that $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^-} f(x)$. We practiced using these definitions.

Wednesday Today we recalled the definition of $f(x)$ as x approaches a real number, and saw that if $f(x)$ is defined on an open interval containing $a \in \mathbb{R}$, then f is continuous at a if and only if $\lim_{x \to a} f(x) = f(a)$. We also recalled the definitions of one-sided limits, which includes approaching ∞ (from the left) and $-\infty$ (from the right). After this, we completed a $\delta - \epsilon$ proof of a limit. Next, we stated the sequential characterization of limits of functions, and the Main Limit Theorem for Functions.

Friday After taking Quiz 8 on limits of functions, we defined the derivative of a function at a point in an open interval of its domain, if it exists. We did several examples of comping the derivative of functions. Finally, we proved that if a function is differentiable at a point, it must also be continuous at the point.

Week 11 October 30–November 3

Read §3.4, 4.1

Homework §3.4: 4–8

Monday We recalled what it means for a sequence of functions on a domain to converge pointwise to a function with the same domain, and to converge uniformly to the function. We proved that $\{(sinx)/n\}$ converges uniformly to $f(x) = 0$ on the entire real line, and proved that the sequence given by $f_n(x) = x^n$ does *not* converge uniformly to its pointwise limit, the piecewise function $f(x) = 0$ if $x < 0$, and $f(1) = 1$. Then we stated and proved an important theorem that replaces the previous proof: If $\{f_n\}$ is a sequence of functions, all continuous on a domain D, then if the sequence converges uniformly to f on D, f must also be continuous on D.

Wednesday Today, we used the theorem stated at the end of class on Monday, and two more, as tests for uniform convergence. We did several examples concluding that s sequence of functions converges uniformly, or does not, to its pointwise limit.

Friday Today was **Midterm 2**.

Week 10 October 23–27

Read §3.3, 3.4

Homework §3.2: 7, 11; §3.3: 1–6; §3.4: 1, 2

Monday We started by recalling the Intermediate Value Theorem (IVT), and using it to prove that the image of a function defined and continuous on a closed, bounded interval is again a closed, bounded interval. We also used the IVT to prove the existence of zeros of a function on an interval. Next, we defined the notion of uniform continuity on a subset of the domain of a function. We noticed that if a function is uniformly continuous on $D \subseteq \mathbb{R}$, then it is continuous on D. We looked at the function $f(x) = \frac{1}{x}$, and guessed that it is not uniformly continuous on $(0, 1]$, and proved that it is uniformly continuous on $[2, 3]$.

Wednesday Today, we started with Quiz 7 on the Intermediate Value Theorem. Then we announced that **Midterm 2 is shifted to Friday, November 3**, and **Group Project 3 on uniform convergence was assigned, due on Wednesday, November 1**. After this, we recalled the definition of uniform continuity. We proved that $f(x) = x^2$ is *not* uniformly continuous on $[0, \infty)]$, but is uniformly continuous on $[0, 10, 000, 000]$. Then we stated a powerful theorem saying that if f is a function that is defined and continuous on a closed, bounded interval, then it is uniformly continuous on that interval.

Friday Today we started by proving that a function that is continuous on a closed, bounded interval is uniformly continuous on the interval. Next, we introduced the notion of a sequence of functions $\{f_n\}$ on some domain $D \subseteq \mathbb{R}$, and defined what it means for such a sequence to converge pointwise on D to some function f with domain D, and to converge uniformly to f. We started looking at several examples to gather intuition on these notions.

Week 9	October 16–20
Read	§3.2, 3.3
Homework	§3.2: 2, 4–6, 8–10
Monday	Fall Break!
Wednesday	Today, we prove that a function that is defined on a closed, bounded interval is bounded on the interval, and attains a minimum and maximum value on the interval. This required us to recall the definition of what it means for a function to be bounded on a subset of its domain, and the supremum/infimum of a function on a subset of its domain. In the proof, we applied the Bolzano Weierstrass theorem, the sequential characterization of continuity, and the fact that a limit of a convergent sequence coming from a closed, bounded interval is itself in the interval. Next, we stated a theorem that the image of a function that is defined and continuous on a closed, bounded interval is again a closed, bounded interval! We will need the Intermediate Value Theorem to prove this next time.
Friday	Today, we took Quiz 6 on monotone sequences and infinite limits. Then we stated and proved the Intermediate Value Theorem. We noticed why the hypotheses are necessary, and stressed that memorizing the hypotheses of named theorems from now one will be very useful, so that we can check easily and accurately whether they apply so that we can take advantage of them.

Week 8 October 9–13

Read §3.2

Homework §3.1: 3–7, 8, 9, 11, 12; §3.2: 1, 3, 5

Monday We started class by handing out Group Project 3 on Cauchy sequences, which is due ~~Wednesday 10/18~~ Friday 10/20 (extended due to Fall Break. Try to take at least some of your break to rest and recharge!). Next we reviewed the $\delta - \varepsilon$ definition of what it means for a function to be continuous at a point a in its domain. We went through several examples in details. Then we stated that a function is continuous on a domain D if it is continuous at every point a in D. We proved that $f(x) = x^2$ is continuous on $\mathbb{R}$, and started proving that $h(x) = \frac{1}{x}$ is continuous on $\mathbb{R} \setminus \{0\}$, which we will finish next time.

Wednesday We started where we left off last time, and proved that $f(x) = \frac{1}{x}$ is continuous on the domain $(0, \infty)$. As an exercise, you can prove that it is continuous on $(-\infty, 0)$ as well, so that it is continuous on its entire natural domain! After this, we stated an alternate characterization of continuity in terms of sequences. We used this characterization, along with the Main Limit Theorem for sequences, to prove that if f and g are both continuous at a point a in both their domains, then cf is continuous at a for any $c \in \mathbb{R}$, $f + g$ and fg are both continuous at a, and if $g(a) \neq 0$, then f/g is also. After this, we stated the fact that if r is a positive rational number, then $f(x) = x^r$ is continuous on its natural domain (which depends on the denominator of r when written in lowest terms!) For practice for the quiz on Friday, try proving this!

Friday We started class by taking Quiz 5 on continuity. After this, we reviewed the sequential definition of continuity, the theorems on continuous functions that can be proved using this characterization (stated last time). We saw that these statements imply that any polynomial is continuous on its domain of all real numbers. Next, we proved, again using the sequential characterization of continuity, that the composition of continuous functions is continuous at a point in its domain, as long as specific continuity assumptions are satisfied for each function. We proved that a complicated function is continuous on its natural domain using this theorem, and the ones recalled earlier today. Finally, we defined what it means for a function to be bounded above/below on a subset of its domain, and the supremum and infimum of the function on such a subset. We stated the theorem that if f is defined and continuous on a closed, bounded interval, then it is bounded on the interval, and achieves both a minimum and maximum value on the interval.

Week 7	October 2–6
Read	§2.5, 3.1
Homework	§2.5: 1–2, 4–8, 9–11; 3.1: 1–2
Monday	We started by reviewing the Monotone Convergence Theorem (MCT) and the definition of an infinite limit, and the following corollary of the MCT: Every monotone sequence has a limit, which may be either a real number or $\pm\infty$. Next, we stated several properties and used them to prove that certain sequences have limit ∞. In doing so, we also proved that if $a > 0$ is a real number, then $n^a \to \infty$. Finally, we defined a closed, bounded interval, and a nested sequence of closed bounded intervals. Then we stated the Nested Interval Property (NIP) that the intersection of a nested sequence of closed, bounded intervals is nonempty.
Wednesday	Today, we recalled the Nested Interval Property, and went over an example and a non-example. We proved the property using the MCT on the sequences of lower and upper endpoints of the intervals. We then defined a subsequence of a sequence, with examples, and stated the Bolzano Weierstrass (BW) theorem, which says that every bounded sequence has a convergent subsequence! We gave another example, an gave the idea for the fact that if a sequence has a (possibly infinite) limit, every subsequence also has this limit. We proved BW using the NIP. Finally, we defined a Cauchy sequence, and stated a theorem staying that a sequence is Cauchy if and only if it converges. We proved one direction of this theorem: every convergent sequence is Cauchy. Next time we will start by proving the other direction.
Friday	We started class by reviewing the definition of a Cauchy sequence, and finishing the proof that a sequence is Cauchy if and only if it converges. We went through an example of showing that a sequence is Cauchy, so therefore converges, though it was not at all obvious what its limit is! Next, we started Chapter 3, and defined what it means for a function from some subset of the real numbers to the real numbers to be continuous at a point in its domain. This definition uses two positive real numbers, ε and δ. We went through several examples of functions continuous at $x = 2$. For $f(x) = x+1$, we drew a picture, and guessed that $\delta = \varepsilon$. The for $g(x) = 3x$, we proved that $\delta = \varepsilon/3$ works. Finally, we started looking at $h(x) = x^2$, which we plan to finish on Monday.

Week 6	September 25–29
Read	§2.4
Homework	§2.4: 1–3, 8–11
Monday	Professor Mat Johnson guest lectured for our class this week, as Emily organized a workshop at Caltech/American Institute of Mathematics. Prof. Johnson began by proving some parts of the Main Limit Theorem, and then start the next section (§2.4) by defining a monotone sequence, going through several examples, and proving the Monotone Convergence Theorem (MCT) today or Friday.
Wednesday	Today was **Midterm 1**. Good luck, all!
Friday	Depending on the pace of lecture on Monday, Prof. Johnson proved the MCT, and then introduce the notion of infinite limits, with plenty of examples.

Week 5	September 18–22
Read	§2.3
Homework	§2.1: 7, 8; §2.2: 2, 3, 5, 6, 8; 2.3: 1–4, 6–9
Monday	We began by recalling the definition of the limit of sequence. We proved that the limit exists, and equals a certain value, for several examples, which required slightly different methods. We also proved that if a sequence converges to a real numbers, the sequence defined by the absolute values of its terms converges to the absolute value of its limit.
Wednesday	Today we recalled the definition of what it means for a sequence to have a limit, and used it to prove that $(-1)^n$ has no limit. Then we proved the Squeeze Theorem for sequences. After this, we stated the Main Limit Theorem, and used it to verify the limit of a complicated-looking rational function.
Friday	We started by taking Quiz 4 on limits of sequences. Then we recalled the Main Limit Theorem and used it to find the limit of a certain sequence. Then we proved that the sequence converges to this limit using the definition of a limit. Finally, we proved that every convergent sequence is bounded, meaning that it both bounded above and bounded below.

Week 4 September 11–15

Read §2.1, 2.2

Homework §1.5: 1–4, 7–9, 13; 2.1: 1–6

Monday We started class by going over Quiz 2. Then we recalled the Archimedean property of the real numbers, and used it to prove that any positive real number x is greater than $1/n$ for some natural number n. We recalled the definition of a nonempty set of real numbers being bounded above, and defined the analogous notion of bounded below. We stated the Completeness Axiom of the real numbers, and used it to prove that any nonempty subset of real numbers that is bounded below has a greatest lower bound. Next, we defined the extended real numbers by adding ∞ and $-\infty$ to the real numbers, and defining inequalities between the elements of these sets. Next, we defined the supremum and infimum of any nonempty subset of real numbers, which is an extended real numbers, and equals the least upper bound or greatest lower bound, respectively, if the set is bounded above or below, respectively. We gave examples of finding the supremum and infimum.

Wednesday Today, we recalled the definition of the supremum and infimum of a nonempty set of real numbers, which is an extended real number. Then we finished an example, finding, with proof, the infimum and supremum of a given set. We did another example, finding the set of all upper bounds for a set, which required polynomial long division. We defined the sum and difference of two sets of real numbers, and stated a theorem relating the suprema and infima of the negative of a set, and these new notions, to the suprema and infima of the original sets. We proved one of the statements. Finally, we stated and proved a theorem characterizing what it means for a real number to greater than or equal to, or less then, the supremum of a set.

Friday Today, we first took Quiz 3 on suprema and infima. Next, we reviewed the Archimedean Principle and its consequence about positive real numbers. After this, we discussed the absolute value and what it means for $|x - a| < \varepsilon$, where a, ε are real numbers and $\varepsilon > 0$. After this, we stated and proved the two parts of the triangle inequality. Finally, we defined a sequence of real numbers, and what it means for a sequence to converge to, or limit to, some real number.

Week 3	September 4–8
Read	§1.4, 1.5
Homework	§1.3: 6–10; §1.4: 1–4, 7
Monday	Labor Day Holiday
Wednesday	First, we took Quiz 2 on induction. After this, we recalled the definition of a commutative ring, and proved some properties about these structures. Then we defined a field, a special kind of commutative ring where every nonzero element has a (multiplicative) inverse. We noticed that the set of rational numbers forms a field, and is in fact a so-called ordered field. We also proved a property about ordered fields.
Friday	We started by proving that $\sqrt{2}$ is not a rational number, though we are familiar with using it as a real number. Next, we defined the set of real numbers via the notion of a Dedekind cuts. We defined the notions of an upper bound and a least upper bound, and proved that the real numbers satisfy the Completeness Axiom. Then we proved that this also satisfies the Archimedean property.

Week 2	August 28–September 1
Read	§1.2, 1.3, 1.4
Homework	§1.1: 9, 12, 14, 15; §1.2: 8–10, 12–14; §1.3: 3, 4
Monday	Today we reviewed the notions of functions between sets from last week, and defined what is means for a function to be one-to-one (also called injective). We proved statements about images and preimages under functions, and found counterexamples to other statements. Then we stated Peano's axioms for the natural numbers and the Principle of Mathematical Induction.
Wednesday	We started with Quiz 1 on sets and functions between sets. We then reviewed the Principle of Mathematical Inductions, and used it to prove 1) that $2 \mid (n^2 + n)$ for every natural number n, and that 2) $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for every natural number n. Then we pointed out that properties of sequences of real number, defined recursively, can be proved via induction.
Friday	We started by proving that a recursively-defined sequence was positive and at most 1 by induction. Then we used a variant of induction to show that $2^n < n!$ for $n \geq 4$. After this, we defined a commutative ring, motivating the fact that these collections of numbers allows us to solve certain algebraic equations. We noticed that the sets of integers, real numbers, and complex numbers are commutative rings using the usual addition and multiplication, as is the set of all polynomials in a variable x with real coefficients.

Week 1 August 21–25

Read §1.1, 1.2

Homework §1.1: 1–8, 10, 11, 13

Monday Today, we started by going over the course website and syllabus. Then we introduced the notion of a set, described and practiced set notation, introduced the notion of the intersection and union of sets.

Before Wednesday, please send me an email with subject line

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MATH 500: Introducing [Your Name]
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and the following info:

- How you would like to be addressed
- Your math background, your relationship with math, what you hope to get from MATH 500, and future goals.
- Anything else you'd like me to know.
- Something extra about you (if you feel comfortable sharing). Pet/hometown photos welcome!

Wednesday We did more examples using set notation and finding intersections and unions of infinitely many sets. We defined the complement of a set A in another set B, and proved that if A and B are both subsets of a set X, then $(A \cup B)^c = A^c \cap B^c$.

Friday Today, we defined a function between sets, the notion of a function being onto/-surjective, and the image and preimage of sets. We proved that if A, B are subsets of a set X, then the complement of $A \cup B$ is the intersection of the complement of A with the complement of B.