Daily Update

MATH 146: Calculus II, Honors

Fall 2019

Class 72: **Thursday, December 12**. We did problems in teams to review the material for the Final Exam.

Class 71: **Wednesday, December 11**. We worked in teams on computations involving the complex numbers, and on the remaining problems on polar coordinates and integration left over from Monday. Our complex number problems are the following:

- Compute (1) $(1+i)^3$, (2) i^{101}.
- If $z = 2+3i$, compute $z^6\overline{z}^7$.
- Find 5 complex numbers for which $\|z\| = 1$.
- Use the Maclaurin series for e^x to compute $e^{2\pi i}$ and $e^{\frac{\pi i}{2}}$.
- Factor the following polynomials completely: x^2-2, x^2+2, x^2+x+1, x^3+x^2+2x+2.

Class 70: **Tuesday, December 10**. Today, we recalled that the **rational numbers** are quotients of integers, and then proved that $\sqrt{2}$ is not rational!

We extend the rational numbers to the larger set of all real numbers. Unfortunately, though, every quadratic equation doesn't have a root that is a real number; for example, any root r of the equation x^2+1 must satisfy $r^2 = -1$.

To remedy this, we define the number i as a square root of -1, so i^2, and also $(-1)^2 = -1$. We immediately see that

$$(x-i)(x+i) = x^2 - i^2 = x^2 - (-1) - x^2 + 1,$$

so i and $-i$ are the roots of x^2+1!

A **complex number** z is a number of the form $a+ib$ for real numbers a and b. We see that $i = 0+1\cdot i$ is a complex number, and so is any real number $a = a+0\cdot i$. The **real part** of a complex number $z = a+ib$ is a, and the **imaginary part** of z is b. We can graph the real numbers in the 2-dimensional **complex plane**, where we label the x-axis as the real axis, and the y-axis as the complex axis, and graph the point (a,b).

The **complex conjugate** of $z = a+ib$ is $\overline{z} = a-ib$. We can see that $z\cdot\overline{z}$ is the real number $(a+ib)(a-ib) = a^2 - i^2b^2 = a^2+b^2$, which we call the **norm** of z, and denote $\|z\|$. Note that the norm of z is the square of its distance from the origin if we graph it in the complex plane!

In order to use complex numbers for arithmetic, we need to add, subtract, multiply, and divide them. We checked by hand that the sum (and so the difference) and product of two complex is again a complex number:

$$(a+ib)+(c+id) = (a+b)+i(c+d)$$
$$(a+ib)(c+id) = ac+iad+ibc+i^2d = (ac-bd)+i(ad+bc).$$

We also want to be able to divide by any nonzero complex number $a+ib$, i.e., when a and b are not both zero. This is the same as the number $\frac{1}{a+ib}$ being a complex number. We noticed that since $(a+ib)(a-ib) = a^2+b^2$, we have that $a-ib = \frac{a^2+b^2}{a+ib}$, so that

$$\frac{1}{a+ib} = \frac{a-ib}{a^2+b^2} = \left(\frac{a}{a^2+b^2}\right) - i\left(\frac{b}{a^2+b^2}\right),$$

a complex number!

Finally, we used the Maclaurin series for e^x to show that $e^{i\theta} = \cos\theta + i\sin\theta$. In particular, this means that $e^{i\pi} = -1$, i.e.,

$$e^{\pi i} + 1 = 0.$$

This equation includes all our fundamental mathematical constants!

Class 69: Monday, December 9. We worked in teams today, finding areas and arc lengths in polar coordinates. We will continue some of these problems tomorrow, and start discussing complex numbers!

Class 68: Friday, December 6. Today, we used calculus to determine a formula for the arc length of a curve! First, we considered a parametric curve given by $x = x(t), y = y(t)$ for $a \leq t \leq b$. We carefully determined (using, in one step, the Mean Value Theorem!) that the arc length equals

$$\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt. \tag{1}$$

We did an example, finding the circumference of a circle of arbitrary radius r, a fact we had always taken for granted!

Next, if $y = f(x)$ is a function of x, we always have parametric equations $x = t, y = f(x)$, so our formula becomes

$$\int_a^b \sqrt{1 + (f'(t))^2}\, dt,$$

where we are considering the arc length for x values from a to b.

We now considered the arc length of the graph of the polar equation $r = 5\cos\theta$. We know that this can be written parametrically as

$$x = r\cos\theta = 5\cos^2\theta y \qquad = r\sin\theta = 5\sin\theta\cos\theta$$

However, the integrand in (1) appeared very complicated. Instead of calculating it completely, we took a step back and noticed that if we have an arbitrary polar curve $r = f(\theta)$, then we have parametric equations

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta.$$

We fully computed the integrand in (1), and cancellation, along with the use of the identity $\sin^2\theta + \cos^2\theta = 1$ allowed us to simplify this as

$$\int_\alpha^\beta \sqrt{(f(\theta)^2 + (f'(\theta))^2}\, d\theta$$

assuming that θ ranges from α to a larger value β! We applied this to our polar curve to obtain the answer 5π; in fact, this is a circle of radius $5/2$! It is important to point out that the entire curve was sketched out for $0 \le \theta \le \pi$, so the bounds of integration were 0 and π (not 2π!).

Class 67: Thursday, December 5. Today we had a group quiz that tested graphing in polar coordinates, moving between polar and Cartesian coordinates, and finding slopes of tangent lines in the polar setting. Then we computed areas in the polar setting by working on 11.4 #7, 11, 13, and 20 in groups.

Class 66: Wednesday, December 4. Today we discussed calculating areas in polar coordinates, and derived the following formula for the area of the region bounded by a polar curve $r = f(\theta)$, and the rays $\theta = \alpha$ and $\theta = \beta$, assuming that $f(\theta) \ge 0$ for $\alpha \le \theta \le \beta$:

$$\frac{1}{2}\int_\alpha^\beta f(\theta)^2\, d\theta.$$

Throughout, we compared this to areas computed via integration using rectangular coordinates.

We did several interesting examples of computing areas of certain regions in polar coordinates. Then we noticed that if $f_1(\theta)$ and $f_2(\theta)$ are polar curves that are positive for $\alpha \le \theta \le \beta$, and $f_2(\theta) \ge f_1(\theta)$ for these values of θ, then the area of the region bounded by $r = f_1(\theta)$, $r = f_2(\theta)$, and the rays $\theta = \alpha$ and $\theta = \beta$, is

$$\frac{1}{2}\int_\alpha^\beta (f_2(\theta)^2 - f_1(\theta)^2) d\theta.$$

We started computing an interesting geometric example that involved realizing a region as the area between curves in polar coordinates. We'll finish it next time!

Class 65: **Tuesday, December 3**. Today, math Ph.D. student Justin Lyle and Amanda Wilkens led the second Investigation Module, on the principle of mathematical induction!

Class 64: **Monday, December 2**. We worked in teams on problems regarding polar equations, in including calculating tangent lines! We sketched the graphs of $r = \sin 2\theta$ and $r = \sin 3\theta$, and then found the points on the first with vertical tangent line, and the points on the second with horizontal tangent line.

Class 63: **Monday, November 25**. Today we continued to discuss polar equations. We determined inequalities in polar coordinates for different regions in the plane, and then derived the formula $r = d\sec(\theta - \alpha)$ for the line with point closest to the origin (d, α), written in polar coordinates. We applied this formula in an example.

We then sketched the graph of $r = \theta$, and then turned to asking where the graph of $r = \sin\theta$ has a horizontal tangent line. We realized that if $r = f(\theta)$ is a polar curve, then it has parametric representation

$$x = f(\theta)\cos\theta \quad y = f(\theta)\sin\theta$$

where $0 \leq \theta \leq 2\pi$ is our parameter. Then

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}.$$

We used this formula to address our question, and then sketched the curve $r = \sin\theta$ to check our answer. Finally, we graphed $r = \sin(2\theta)$ in groups.

Class 62: **Friday, November 22**. We introduced and studies the notion of polar coordinates. Every point in the plane can be represented in Cartesian, or "rectangular," coordinates as $P = (x, y)$, but can be represented in **polar coordinates** (r, θ), where r is the distance of P from the origin, and θ is the angle between the positive x-axis and the line passing through P and the origin.

We plotted several points in polar coordinates, and noticed that $(r, \theta) = (r, \theta + 2\pi k)$ for any integer k. By convention, if $r > 0$, $(-r, \theta)$ is the reflection of (r, θ) through the origin, i.e., $(-r, \theta) = (r, \theta + \pi)$. Any point of the form $(0, \theta)$ is the origin.

Given a point (r, θ) in polar coordinates, we computed geometrically that its representation in (x, y) in Cartesian coordinates is given by $x = r\cos\theta$ and $y = r\sin\theta$.

On the other hand, given (x, y) in Cartesian coordinates, we showed that its polar coordinates (r, θ) satisfy

$$r^2 = x^2 + y^2 \text{ and } \tan\theta = y/x.$$

However, we pointed out, and illustrated this point with an example, that θ does not necessarily equal $\arctan(y/x)$–we must consider the quadrant that the point lies in in order to find its polar angle.

We noticed that the graph of the polar equation $r = 5$ is the circle of radius 5 centered at the origin (and so is $r = -5$!). On the other hand, the polar equation $\theta = \pi/4$ is the line through the origin of slope 1.

We graphed the polar equation $r = 2\cos\theta$ by plotting points from $\theta = 0$ to 2π in increments of $\pi/4$, and noticed that it looked like a circle of radius 1 centered at the Cartesian point $(1, 0)$. Then we converted its equation, using the identities determined earlier, to the Cartesian equation $(x-1)^2 + y^1 = 1$, verifying our conjecture! In this case, the graph is symmetric about the x-axis, since whenever (r, θ) is on the graph, so is $(r, -\theta)$; we can check this using the polar equation!

Next, we plotted points to sketch the graph of the polar equation $r = 1 + \sin\theta$, which is an example of a *cardiod*.

Class 61: **Thursday, November 21**. We started class with a quiz on applying vector geometry, which involved dot products and cross products.

We then turned to estimating the probability that a random variable is in the interval $[1, 4]$ using the standard normal distribution presented last time, which means that we estimated

$$\frac{1}{\sqrt{2\pi}}\int_1^4 e^{-x^2/2}\,dx$$

by using Taylor series, and the estimate for partial sums of alternating series.

Class 60: **Wednesday, November 20**. The first part of class today was focused on applications of vector geometry to physics. First, we recalled how to find the force on cables from which a mass is hanging.

After this, we turned to the use of the cross product to physics. We computed the force on a moving charge with velocity $\mathbf{v}$ meters per second in a uniform magnetic field $\mathbf{B}$ (in teslas) as $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$ newtons, where $q = 1.6 \cdot 10^{-19}$ coulombs. Then we computed the **torque** about the origin due to a force $\mathbf{F}$ newtons acting on an object with a position vector $\mathbf{r}$ (in meters) as $\tau = \mathbf{r} \times \mathbf{F}$ N-m.

After this, we turned to the notion of probability, and mentioned what the probability of two of us having the same birthday. We introduced the notion of a **probability density function** $p(x)$, a non-negative continuous function such that $\int_a^b p(x)\,dx$ equals the probability $P(a \le X \le b)$ that a certain quantity X (called a **random variable**) is in the interval $[a, b]$. In particular, $\int_{-\infty}^{\infty} p(x)\,dx = 1$.

We gave a few examples, and then showed that the function $p(x) = \frac{1}{1+x^2}$ is not a probability density function, but that $q(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$ is one.

We introduced the **standard normal** density function $p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, and asked how we would use it to find (or estimate) the probability that a random value lies in a certain interval. The answer: Taylor series!

Class 59: **Tuesday, November 19**. We worked in teams on problems involving equations in three dimensions, including lines and planes.

Class 58: **Monday, November 18**. Today we discussed planes in $\mathbb{R}^3$. We noticed that a plane is determined by 1) a point $P_0 = (x_0, y_0, z_0)$ on the plane, and 2) a **normal** vector $\mathbf{n} = \langle a, b, c\rangle$ to the plane; i.e., a vector that is perpendicular to any vector on the plane. We derived equations for such a plane, meaning that a point (x, y, z) is on the plane if and only if it satisfies any one of the following equations:

- (Scalar equation 1) $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$
- (Scalar equation 2) $ax + by + cz = d$
- (Vector equation) $\mathbf{n} \cdot \langle x, y, z\rangle = \mathbf{n} \cdot \overrightarrow{P_0P}$

where $d = ax_0 + by_0 + cz_0 = \mathbf{n} \cdot \overrightarrow{0P_0}$.

We found the equation of a given plane in several examples. In the first, we argued that the given plane must be the xy-plane, and obtained the equation $2z = 0$, which is equivalent to the traditional equation $z = 0$. We noticed that given the equation of a plane, we can scale both sides of the equation by a nonzero scalar to find another equation for the same plane. We then showed that all planes with equations of the form

$$ax + by + cz = d$$

are parallel to one another, as the value of d ranges through all real numbers. The unique plane in this family that contains the origin is $ax + by + cz = 0$.

Finally, we pointed out that given three points that are not **collinear**, meaning that they do not all lie on any one line, then there is a unique plane containing the three points. We then started working through an example in finding the equation of such a plane, given three points. As an exercise, we were asked to check that the lines are not collinear, and find the final answer. We needed to use the cross product to complete the problem!

Class 57: **Friday, November 15**. We finished computing the volume of a parallelepiped defined by nonzero vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^3$, which equals the norm of $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$; this vector is often called the **vector triple product**. We realized this vector as a determinant.

After a short quiz on vector geometry, we worked in teams on problems related to areas and volumes using cross products.

Class 56: **Thursday, November 14**. We continued our discussion of the cross product, reviewing the important properties that we've already discussed We noticed the striking property

$$\mathbf{w} \times \mathbf{v} = -\mathbf{v} \times \mathbf{w}$$

holds. In particular, taking cross products is *not* commutative! We also described several other properties of the cross product: the cross product of a vector with itself is the zero vector, the cross product of two vector is zero if and only if one is a scalar multiple of the other (or one vector is zero), we can "pull out scalars" from the cross product, and the cross product and vector addition satisfy the distributive law.

From here, we investigated the cross product among pairs of the standard unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, and applied it in an example.

We then determined how to compute the cross product of two vectors when written as linear combinations of $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$.

We then showed that the area of the parallelogram defined by two vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$ equals $\|\mathbf{v} \times \mathbf{w}\|$!

This means that the triangle obtained by connecting their tails has area $\frac{1}{2}\|\mathbf{v} \times \mathbf{w}\|$.

After this, we started investigating the volume of a parallelepiped (3-dimensional prism) defined by three vectors in $\mathbb{R}^3$. We fill finish this computation next time!

Class 55: **Wednesday, November 13**. Today, Amanda Wilkens ed the class in working in teams through problems involving the dot product and cross product of vectors.

Class 54: **Tuesday, November 12**. Guest lecturer Professor Daniel Hernández discussed the *cross product* of two vectors in $\mathbb{R}^3$ with the class.

The definition of a cross product involves the notion of *determinants.* A 2×2 matrix has the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and its **determinant** is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

We then asked the fundamental question: Why is this called a determinant? What is it determining? To answer this, we proved the following **Theorem**: The determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is zero if and only if one of $\langle a, b\rangle$ and $\langle c, d\rangle$ is a multiple of the other.

The determinant of a 3×3 matrix $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

Now, the **cross product** of $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3\rangle$ is defined to be the *vector*

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \mathbf{i} + \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \mathbf{k}.$$

In particular, the cross product of two vectors is a **vector**, while we know that the dot product of two vectors is a *scalar*!

We did some examples, and then gave a geometric description of the cross product $\mathbf{v} \times \mathbf{w}$ of nonzero vectors: It is the unique vector satisfying the following three properties:

1. $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.
2. $\mathbf{v} \times \mathbf{w}$ has length $\|\mathbf{v}\|\|\mathbf{w}\| \sin\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.
3. The vectors $\mathbf{v}, \mathbf{w}, \mathbf{v} \times \mathbf{w}$ form a right-handed system. That is, the direction of $\mathbf{v} \times \mathbf{w}$ is given by the right-hand rule.

We elaborated on the right-hand rule, and presented a bunch of (poorly-drawn) examples that showed how to figure out the direction of $\mathbf{v} \times \mathbf{w}$. We used this geometric description to find the cross product in an example, rather than using the formula, to verify that $\langle 2, 0, 0\rangle \times \langle 0, 1, 1, \rangle = \langle 0, -2, 2\rangle$, a formula we had verified earlier in class.

Class 53: **Monday, November 11**. Today we introduce the **dot product** of two vectors in $\mathbb{R}^3$: Given $\mathbf{v} = \langle v_1, v_2 v_3\rangle, \mathbf{w} = \langle w_1, w_2, w_3\rangle \in \mathbb{R}^3$,

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3.$$

In particular, the dot product of vectors is a scalar!

We did some examples, and noticed that the dot product of a vector with itself is the square of its length: $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$. We also noted some other properties of the dot product: the dot product of any vector with the zero vector equals zero, the operation is commutative, we can "pull out scalars," and the distributive law holds with respect to the dot product and vector addition.

By convention, the angle between two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ is $0 \leq \theta \leq \frac{\pi}{2}$, and in fact,

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta.$$

We noticed that this means that

$$\theta = \arccos\left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right).$$

We say two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ are **perpendicular** or **orthogonal**, and write $\mathbf{v} \perp \mathbf{w}$, if the angle between them is $\theta = \frac{\pi}{2}$, which we determined is the same as saying that their dot product is zero! We checked that the standard basis vectors are pairwise perpendicular, and then investigated in other examples whether given vectors are perpendicular to one another.

We also noticed that the angle between vectors is *obtuse* if their dot product is negative, and is *acute* if it is positive.

We set out to find an orthogonal vector to a given vector, and were able to find many. Then we defined the **projection** of a nonzero vector $\mathbf{v}$ onto a vector $\mathbf{u}$,

$$\mathbf{u}_{||\mathbf{v}} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|}\right)\mathbf{e}_{\mathbf{v}}.$$

This vector has a natural graphic description, which we illustrated. In particular, $\mathbf{u}_{||\mathbf{v}}$ is parallel to $\mathbf{u}$.

Finally, we wrote the **decomposition** of a vector $\mathbf{u}$ in terms of a nonzero vector $\mathbf{v}$:

$$\mathbf{u} = \mathbf{u}_{||\mathbf{v}} + \mathbf{u}_{\perp\mathbf{v}}.$$

We solved for $\mathbf{u}_{\perp\mathbf{v}}$ in terms of the other vectors, and checked that it is, in fact, perpendicular to $\mathbf{u}$! Hence and vector can be written as the sum of a vector parallel to it, with a vector that is perpendicular to it.

Class 52: **Friday, November 8**. Today we worked in groups on many problems regarding vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, and equations in $\mathbb{R}^3$.

Class 51: **Thursday, November 7**. We discussed 3-dimensional space, $\mathbb{R}^3$, today. We talked about the right-hand rule convention. Then we discussed distances and graphs in $\mathbb{R}^3$, including formulas for planes, spheres, and cylinders. We also discussed vectors in $\mathbb{R}^3$, parameterizations of lines, and checking properties of lines.

Class 50: **Wednesday, November 6**. Today, we started our first Investigation Module, led by mathematics Ph.D. student Amanda Wilkens! The module is on the formal mathematical definition of the limit of a sequence. Amanda and another Ph.D. student, Justin Lyle, created the module.

Class 49: **Tuesday, November 5**. We started class by defining a **linear combination** of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in $\mathbb{R}^2$: a vector of the form

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n,$$

where all of the λ_i are real numbers, or *scalars*.

We wrote a given vector as a linear combination of two other vectors by setting up a system of equations obtained by setting the components equal. We also investigated how to visualize the linear combination of two vector graphically.

We noticed immediately that every vector can be written as a linear combination of the vector $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$.

We conjectured that given a vector $\mathbf{v}$, there is another vector with the same direction, with length 1; i.e., a **unit vector**. We determine that this vector is unique and equals

$$\mathbf{e}_{\mathbf{v}} = \frac{1}{\|\mathbf{v}\|} \cdot \mathbf{v}.$$

Note that $\frac{1}{\|\mathbf{v}\|}$ is just a scalar!

We drew a picture, and noticed that $\mathbf{e_v}$ has its endpoint on the unit circle, so that by this fact, and the equation above,

$$\mathbf{v} = \|\mathbf{v}\|\mathbf{e_v} = \|\mathbf{v}\|\langle \cos\theta, \sin\theta\rangle$$

assuming that θ is the angle between $\mathbf{v}$ and the positive x-axis.

Finally, using vector arithmetic, we set up and solved a problem of finding the force on each of two ropes, with a mass hanging from them.

Class 48: **Monday, November 4**. Today we introduced **vectors** in the real 2-dimensional plane $\mathbb{R}^2$. A vector $\mathbf{v}$ in $\mathbb{R}^2$ has a base point, and an endpoint, so has a direction and **length**, or **magnitude**, denoted $\|\mathbf{v}\|$. Note that the **zero vector**, denoted $\mathbf{0}$, has length 0. Two vectors are **parallel** if the lines extending from them are parallel. We say that two vectors are **equivalent** if they are translations of one another, so have the same length and direction.

If a vector $\mathbf{v} \in \mathbb{R}^2$ has basepoint $P = (a_1, b_1)$ and endpoint $Q = (a_2, b_2)$, the **x-component** of $\mathbf{v}$ is $a_2 - a_1$, and its **y-component** is $b_2 - b_1$. In other words, if $\mathbf{v}$ is translated to have basepoint the origin O, these are the x- and y-coordinates of the endpoint. In this notation, $\|\mathbf{v}\| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2}$

We did an example, deciding that two vectors are not equivalent by verifying that one of their components were not equal.

We **add** two vectors $\mathbf{v}$ and $\mathbf{w}$ by adding their components. Graphically, $\mathbf{v} + \mathbf{w}$ is the vector obtained in the following way: put a translate of $\mathbf{w}$ at the endpoint of $\mathbf{v}$, and take the vector with the basepoint of $\mathbf{v}$ and the endpoint of the translate of $\mathbf{w}$. (Here, the roles of $\mathbf{v}$ and $\mathbf{w}$ can be switched!)

We also discussed how to **scale** a vector by a real number λ: $\lambda\mathbf{v}$ is the vector whose components are both scaled by λ. Graphically, the new vector has length $|\lambda|$ times the original one, and points in the same direction as the original one precisely if λ is positive.

Finally, we noted that vector addition is *commutative* and **associative**, and scalar multiplication and addition together satisfy the *distributive* law.

Class 47: **Friday, November 1**. We started class by reviewing how to determine power series representations of certain functions: The first is to start with known power series representations for given functions, valid for inputs in their intervals of convergence, and then manipulate them by substituting, adding, subtracting, multiplying, differentiating, or integrating them to get a new power series. We worked through the last problems from yesterday's group work to see examples of this.

The other method is to find the Taylor series centered about an x-value. The issue here is that the Taylor series need not always agree with the function value in its interval of convergence! We had not seen an example of this, which is the main goal of class today. The function we studied is

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

We determined that this function is continuous, and that as $x \to \pm\infty$, $f(x) \to 1$. We sketched a graph of $f(x)$, and then worked toward computing its Maclaurin series. We noticed that f is differentiable, and after taking some derivatives for $x \neq 0$, we noticed that for $x \neq 0$,

$$f^(n)(x) = \frac{P(x)e^{e^{-\frac{1}{x^2}}}}{x^r}$$

for some polynomial $P(x)$ and integer $r \geq 2$! In fact, this will allow us to show that all derivatives of f exist at $x = 0$, and equal 0! In particular, the Maclaurin series for f is the power series where all coefficients are 0, which converges to 0 for all x. However, it only agrees with $f(x)$ for $x = 0$!

Class 46: **Thursday, October 31**. Today we introduced the **binomial series**, which converges to the function $(1+x)^a$ for $-1 < x < 1$: Given a real number a and an integer $n \geq 0$, we define

$$\binom{a}{n} = \frac{a(a-1)(a-2)\cdots(a-n+1)}{n!}$$

where if $n = 0$, we interpret this value as $\binom{a}{0} = 1$. Then for $|x| < 1$, the binomial series is

$$(1+x)^a = 1 + \frac{a}{1!}x + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \binom{a}{n} x^n.$$

In groups, we then found the binomial expansion where $a = \frac{1}{2}$, i.e., a power series representation for $\sqrt{1+x}$ for $|x| < 1$. We finished the last two problems from last time, and then worked on #46 and #65 in section 10.7.

Class 45: **Wednesday, October 30**. We worked in teams today in solving problems on power series, especially related to Taylor series:

- Find the first five terms of a power series representation for $f(x) = e^x \sin x$. For what x values does the power series represent $f(x)$?
- Find the Maclaurin series for $\ln(1+x)$ and its radius of convergence.
- Find a power series expansion for $x^4 - x^3 + 16x^2 - 11 + \ln(1+x)$.
- Find a power series expansion for $f(x) = \frac{\sin\left(\frac{x}{2}\right)}{3}$. For what x values does the power series represent $f(x)$?

- Determine what functions the following power series represent:
  - $1 + x^3 + \frac{x^6}{6!} + \frac{x^9}{3!} + \frac{x^{12}}{4!} + \cdots$
  - $x^4 - \frac{x^{12}}{3} + \frac{x^{20}}{5} - \frac{x^{28}}{7} + \cdots$

Class 44: Tuesday, October 29. Today, we reviewed how to obtain a power series representation of a function using the properties of a geometric series, for the functions

$$f(x) = \frac{1}{1+x^2} \text{ and } g(x) = \frac{1}{1-2x}.$$

Note that the radius of convergence of power series for the first series is 1, while for the second, it is 1/2!

We noticed that we can't use this idea to find a power series representation for $f(x) = \cos x$, so we use a Taylor series. Yesterday, we derived the Maclaurin series for this function, and we wrote this as

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

We showed, using the theorem from last time, that since all derivatives of $\cos x$ are bounded in absolute value by 1, this series represents $\cos x$ for all values of x!

We noticed that we could do the same for $\sin x$ to get the representation

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

for all real numbers x

We went through all the details in finding the Maclaurin series for $f(x) = x^3$, which turned out the be (unsurprisingly)

$$0 + 0 \cdot x + 0 \cdot x^2 + 1 \cdot x^3 + 0 \cdot x^4 + 0 \cdot x^5 + \cdots$$

We also know the Maclaurin series for e^x, and we used this to construct a power series expansion for $g(x) = xe^x$.

From here, we turned to the question of whether we can use the expansion for e^x to get one for e^{x^2}: The answer was *yes*, by substituting x^2 for x. However, this does not work for $e^{\sin x}$, since we don't get a power series after substituting! On the other hand, we noticed that it may be possible to find a power series representation for $e^x \sin x$ via "infinite distributing"–we'll work on this next time in groups!

Class 43: Monday, October 28. Today we recalled the definitions of the radius/interval of convergence of an infinite series, and the definition of a power series centered at c. A Taylor series centered at $c = 0$ is called a **Maclaurin series**.

We stated a **theorem** that if a function $f(x)$ is represented by a power series for all x in the interval $I = (c - R, c + R)$ for some $R > 0$, then this power series *must be the Taylor series*. However, it is not necessarily the case that the Taylor series must converge to the function's value!

We found the Maclaurin series for $f(x) = e^x$:

$$1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

and found that its radius of convergence is ∞. However, we did not show that the series represents the function for all real numbers x.

In general, there is no guarantee that the Taylor series converges to $f(x)$, even if it converges! In fact, the Taylor series converges to $f(x)$ if and only if $\lim_{k\to\infty} R_k(x) = 0$, where $R_k(x)$ is the *k-th remainder series*:

Given a Taylor series

$$T(x) \sum_{n=1}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots ,$$

the **k-th Taylor polynomial** is

$$\sum_{n=1}^{k} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_k(x-c)^k$$

and the **k-th remainder** is

$$R_k(x) = T(x) - T_k(x) = \sum_{n=k}^{\infty} a_{k+1}(x-c)^{k+1} + a_{k+2}(x-c)^{k+2} + \cdots$$

We stated the following **theorem**: If there is some number $K >$ for which

$$|f^{(k)}(x)| \leq K$$

for all $k \geq 0$ and all x in the interval $I = (c - R, c + R)$, then f is represented by its Taylor series in the interval I.

We wrote out the first few terms of the Maclaurin series for $f(x) = \cos x$, and then noticed that the condition in the theorem above is satisfied for $K = 1$. As homework, find an expression for the general term of this Maclaurin series!

Class 42: **Friday, October 25**. Today we had another guest lecture by Professor Hernández. We recalled the key ideas from last lecture, and in groups, we were asked to find a power series expansion and radius of convergence for $\frac{1}{(3-x)^2}$ with center $c = \pi$. Students worked in groups together to solve this, and everyone had the right approach: First, find the power series expansion for $1/(3 - x)$ with center $c = \pi$, and then apply term-by-term differentiation to get the expansion for $1/(3-x)^2$. After this, we presented an example of how to use term-by-term integration to derive a power series expansion for $\tan^{-1}(x)$; this time, the key observation was that the derivative of $\tan^{-1}(x)$ is $1/(1+x^2)$, which we can find a power series expansion for. After that, we applied term-by-term integration, and solved for the constant of integration that popped up.

After this, we asked the following **Question**: Is there any pattern to the terms a_n that have popped up when we've been writing $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ the past couple of days? One pattern is clear: If we let $x = c$ on both sides, then we get $f(c) = a_0$. In fact, by applying term-by-term differentiation to the power series, we were able to prove the following **Theorem**: If f can be written as a power series

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

with radius of convergence R, then

$$a_n = \frac{f^{(n)}(c)}{n!}$$

where $f^{(n)}$ denotes the n-th derivative of f. Motivated by this, we then presented the **Definition**: The Taylor series of a function f with center c is the power series

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

The previous theorem then says that if f can be written as a power series with center c, then that power series **must** be the Taylor series of f! We then revisted an earlier example to make sure that this was the case.

Class 41: **Thursday, October 24**. Today we had another guest lecture by Professor Hernández. We started off by recalling the key points from last lecture, especially the power series expansions $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ and $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$, both of which converge for $|x| < 1$. Using these two series, we were able to describe a number of other functions in terms of power series, and also compute the radius of convergence for these series. For instance, we showed that

$$\frac{1}{2+x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{n+1}}$$

and that this series had radius of convergence $R = \sqrt{2}$. After going over more examples like this, we then shifted our attention to figuring out how to find power series descriptions with center different than 0. For instance, we showed that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{(-1)^n (x-4)^n}{3^{n+1}}$$

which is a power series centered at 4 with radius of convergence $R = 3$.

After this, we stated an important theorem that tells us we can differentiate and integrate power series term-by-term. **Theorem**: Consider $F(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$, a power series centered at c with radius of convergence R. If x lies within the interval of convergence $(c-R, c+R)$, which we interpret to be all of $\mathbb{R}$ when $R = \infty$, then the following hold.

1. $F'(x) = \sum_{n=0}^{\infty} na_n(x-c)^{n-1}$.
2. $\int F(x)dx = \sum_{n=0}^{\infty} \frac{a_n(x-c)^{n+1}}{n+1}$.

We pointed out how these would be trivial if we had a finite sum instead of a series, and we used the formula for $F'(x)$ to get a power series formula for $\frac{1}{(x-1)^2}$. We'll show how to use the formula involving integrals tomorrow. We concluded the lecture with Quiz 8.

Class 40: **Wednesday, October 23**. Today, the class worked on calculating power series and radii of convergence with Professor Międlar.

Class 39: **Tuesday, October 22**. Today we had a guest lecture by Professor Hernández. Today's lecture was all about **power series**. Recall that a power series is an infinite series of the form

$$F(x) = \sum_{n=0}^{\infty} a_n(x-c)^n.$$

We call the term c in this expression the **center** of the power series. After giving this example, we then went over a bunch of examples, and in each of these examples, we plugged in various x values to see that for some values of x, the power series $F(x)$ converges, and for other values of x, it diverges. Indeed, if $F(x) = \sum_{n=0}^{\infty} \frac{x^n}{3^n}$, we saw that $F(2)$ converged, but that $F(\pi)$ diverged. In fact, for this same power series, we saw that $F(x)$ converged to $\frac{3}{3-x}$ if $|x| < 3$, i.e., if $-3 < x < 3$. We then stated the following result.

Theorem: Every power series $F(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ has a *radius of convergence* R. The radius of convergence R either satisfies $0 \leq R < \infty$ or $R = \infty$.

1. If $0 \leq R < \infty$, then $F(x)$ converges absolutely for every x satisfying $|x-c| < R$, i.e., for every $c - R < x < c + R$.
2. If $R = \infty$, then $F(x)$ converges absolutely for every real number x.

As a consequence of this theorem, we noted that to completely describe the set of *all* x-values for which $F(x)$ converges, we must calculate the radius of convergence R (e.g., using the Ratio Test), and then test for convergence at any endpoints. We then did this for the power series $\sum_{n=0}^{\infty} \frac{(-1)^n(x-5)^n}{4^n n}$ to see that it converged for every $1 < x \leq 9$. Next, we applied this logic to the power series $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ to see that it converged (absolutely) for every real number x. In other words, in this case, the radius of convergence was infinity.

Class 38: **Monday, October 21**. The class worked on problems involving the ratio and root tests with Professor Międlar.

Class 37: **Friday, October 18**. Motivated by series that don't have all non-negative terms, so that many of the tests we've developed don't apply, we presented the **ratio test**: Suppose that

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

exists, or that the sequence $\{\left|\frac{a_{n+1}}{a_n}\right|\}$ diverges to infinity.

- If $\rho < 1$, then $\sum a_n$ converges absolutely.
- If $\rho > 1$ or $\left|\frac{a_{n+1}}{a_n}\right|$ diverges to infinity, then $\sum a_n$ diverges.
- If $\rho = 1$, then the test is inconclusive (the series can have any possible convergence behavior).

We applied this test to the series $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ and $\sum_{n=1}^{\infty} \frac{.0001^n}{n^2}$ to determine their convergence, and noticed that the test is inconclusive for both $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$, despite them having different convergence behavior.

We then approached the series $\sum_{n=1}^{\infty} \left(\frac{n}{5n-3}\right)^n$ and looked into applying the ratio test; the expression $\left|\frac{a_{n+1}}{a_n}\right|$ was pretty complicated Instead of pursuing this further, we presented the **root test**: Suppose that

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

exists, or that the sequence $\{\sqrt[n]{|a_n|}\}$ diverges to infinity.

- If $L < 1$, then $\sum a_n$ converges absolutely.
- If $L > 1$ or $\left|\frac{a_{n+1}}{a_n}\right|$ diverges to infinity, then $\sum a_n$ diverges.
- If $L = 1$, then the test is inconclusive (the series can have any possible convergence behavior).

We used this to determine that the series $\sum_{n=1}^{\infty} \left(\frac{n}{5n-3}\right)^n$ converges, but that $\sum_{n=1}^{\infty} \left(\frac{7n}{5n-3}\right)^n$ diverges. Moreover, the root test applied to $\sum_{n=1}^{\infty} \left(\frac{5n}{5n-3}\right)^n$ is inconclusive!

Finally, we wrote several different series on the board, and discussed some basic ideas on how to determine which test (or combination of tests) to use.

Class 36: Thursday, October 17. Today, we reviewed what it means for a series to be **absolutely convergent**, and reminded ourselves that any absolutely convergent series is automatically also convergent. We call a series **conditionally convergent** if it is convergent but not absolutely convergent. Our theory so far implies that every series can be classified as one of the following:

- absolutely convergent, or
- conditionally convergent, or
- divergent (in which case, the series of absolute values of the terms also diverges).

We then discussed the **alternating series test (AST)**: If the sequence $\{b_n\}$ is positive, decreasing, and has limit 0, then the alternating series

$$S = \sum_{n=1}^{\infty}(-1)^n b_n = b_1 + b_2 + b_3 - b_4 + \cdots$$

converges. Moreover, when S is approximated by the partial sum S_N, the error is less than the first omitted term b_{N+1}; i.e.,

$$|S - S_N| < b_{N+1}.$$

We applied this to show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{2n}}$ and $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ both converge. We also estimated series using the second part of the AST statement.

Finally, we worked in groups on problems related to this material.

Class 35: Wednesday, October 16. We presented the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\cdot 3^n}$ to remind ourselves of the tests we have discussed that can be used to decide whether (eventually) **nonnegative** series converge or diverge (i.e., the terms of the series are eventually all non-negative). Both the **integral test** and the **comparison test** ended up working to show that this series converged, but in the comparison test, we needed to compare to the series $\sum_{n=1}^{\infty} \frac{1}{3^n}$ (which converges since it is a geometric series with $r = \frac{1}{3} < 1$) rather than $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ (which diverges by the p-series test).

Through our review of the comparison test, we discussed in detail the fact that the convergence or divergence of a series does not depend on the first terms, so we may use some tests even when the indices of the series do not start at the same integer.

We then presented the example $\sum_{n=1}^{\infty} \frac{n^3}{n^5-n-1}$, and noticed that the series is positive after the first term. However, though we thought that it should behave similarly to $\sum_{n=1}^{\infty} \frac{n^3}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges by the p-series test. However, the comparison was the wrong direction, i.e.,

$$a_n = \frac{n^3}{n^5 - n - 1} > \frac{n^3}{n^5} = \frac{1}{n^2} = b_n$$

so we could not use the comparison test.

However, we can use another test called the **limit comparison test**: Suppose that $\sum a_n$ and $\sum b_n$ are series with (eventually) nonnegative terms, and let

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

Then

- If $L > 0$ is finite, then $\sum a_n$ converges if and only if $\sum b_n$ converges.
- If $\frac{a_n}{b_n} \to \infty$ and $\sum a_n$ converges, then $\sum b_n$ converges.
- If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

We found that $L = \lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$ in our example above, so that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^3}{n^5-n-1}$ converges since $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ does.

We used the limit comparison test to find that $\sum_{n=1}^{\infty} \infty \frac{e^n+n}{e^{2n}-n^2}$ converges (we needed L'Hôpital's rule!), and gave the **exercise** to show that $\sum_{n=1}^{\infty} \sqrt{3n^2+9}$ diverges by comparing it to either $\sum_{n=1}^{\infty} \frac{1}{n}$ or $\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n^2}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{3}n}$.

Even though the limit comparison test looks more powerful than the original comparison test (which we will now sometimes call the **direct comparison test**), the direct comparison test can be used in some cases when the limit version does not apply; e.g., sometimes when the terms of a series are very complicated, but admit a comparison with simpler terms.

From here, we defined a series (now arbitrary, i.e., possibly having some negative terms) to be **absolutely convergent** if the series $\sum |a_n|$ converges.

We showed that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent since $\left|\frac{(-1)^{n-1}}{n^2}\right| = \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. However, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is *not* absolutely convergent.

In fact, it is a **theorem** that an absolutely convergent is always convergent! Hence we know that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ converges, though none of our series convergence tests apply since it is not a nonnegative series.

We defined an **alternating series** as one that alternates between positive and negative terms, our example $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} + \cdots$ is an alternating series.

Class 34: **Friday, October 11**. We started class with a short quiz on what it means for an infinite series to converge. Next, we focused on looking at series with positive or nonnegative terms. We call a series

$$\sum_{n=1}^{\infty} a_n$$

positive if all $a_m > 0$, and **nonnegative** if all $a_n \geq 0$. Note that throughout class today, we will write series to begin at index $n = 1$, but we can modify statements to deal with those starting at other indices.

We noticed that in these cases, if $\{S_N\}$ is the sequence of partial sums, then either:

1. S_N is bounded above, in which case the series converges, or
2. S_N is not bounded above, in which case it diverges to infinity.

From here, we presented the **integral test** for the converges of infinite series: Let $a_n = f(n)$ for a function f that is nonnegative, decreasing, and continuous for all $x \geq 1$. Then

1. If $\int\limits_1^\infty f(x)\,dx$ converges, then the series $\sum\limits_{n=1}^\infty a_n$ converges.
2. If $\int\limits_1^\infty f(x)\,dx$ diverges, then the series $\sum\limits_{n=1}^\infty a_n$ diverges.

We gave arguments involving Riemann sums that explain why these hold.

As a consequence of the integral test, we have the p**-series test** for series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \textbf{ converges if } p > 1, \text{ and diverges if } p \leq 1.$$

As a consequence, we concluded that the so-called **harmonic series** $\sum\limits_{n=1}^\infty \frac{1}{n}$ diverges!

Finally, we pointed out that from the integral test for series, along with the comparison test for improper integrals, we obtain a **comparison test** for series: If $0 \leq a_n \leq b_n$ for n large enough, then regardless of where the indices start,

1. If $\sum b_n$ converges, then $\sum a_n$ converges, and
2. If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Finally, we worked in teams to determine the convergence or divergence of different series, using the material learned today, and from previous classes. When the series converged, we tried to find its value.

Class 33: **Thursday, October 10**. Today we determined which **geometric series** $\sum\limits_{n=0}^\infty cr^n$ converge and which diverge by determining a formula for the N-th partial sum of the series, $S_N = \frac{c(1-r^{N+1})}{1-r}$ (as long as $r \neq 1$). If $|r| < 1$, then the series converges and equals $\frac{c}{1-r}$, but if $|r| > 1$, then the series diverges. We used this to show that $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$ in a new way, and then we used it to show that $\sum\limits_{n=0}^\infty \frac{3+4^n}{7^n}$ converges, and find its value.

Next, we stated the **divergence test** for infinite series, which says that if $\lim_{n\to\infty} a_n \neq 0$, then an infinite series $\sum a_n$ (starting at any integer index) diverges! We saw that this means $\sum\limits_{n=1}^\infty \frac{7n}{10n+1}$ and $\sum\limits_{n=1\infty} (-1)^{n-1}$ diverge. (The latter also diverges since it is a geometric series!)

Class 32: Wednesday, October 9. Today we started by showing that $a_n = \sqrt[3]{n+1} - n$ diverges by showing that it is not bounded below, and then we showed that the sequence

$$a_1 = \sqrt{2},\ a_2 = \sqrt{2\sqrt{2}},\ a_3 = \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

is bounded above by 2, and increasing, so it converges!

After this, we introduced the notion of an **(infinite) series**, denoted $\sum\limits_{n=1}^{\infty} a_n$ or $\sum\limits_{n=k}^{\infty} a_n$ for any integer k. For instance, if $a_n = \frac{1}{2^n}$, then we are considering

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

Our goal for today is to define what this *means*.

Given a sequence $\{a_n\}$ indexed by $n \geq 1$, we define the associated **sequence of partial sums** $\{S_N\}$ as:

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ S_N &= a_1 + a_2 + \cdots + a_N \end{aligned}$$

i.e., to find S_N, we add up all terms of the original sequence until $n = N$.

We computed in our example where $a_n = \frac{1}{2^n}$ that $S_N = 1 - \frac{1}{2^N}$.

Next, we gave the following definition: An infinite series $\sum\limits_{n=1}^{\infty} a_n$ **converges** if the limit of the sequence of partial sums exists (and equals a finite number), i.e., $\lim\limits_{N\to\infty} S_N = S$ for some real numbers S. In this case, we say that the series equals S, and write:

$$\sum_{n=1}^{\infty} a_n = S.$$

If the sequence of partial sums grows without bounds, we say that $\sum\limits_{n=1}^{\infty} a_n$ **diverges to infinity**, and if $\lim\limits_{N\to\infty} S_N$ does not exist, we say that $\lim\limits_{N\to\infty} S_N = S$ **diverges**.

We used this definition to show that $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges and equals 1. Then we found that $\sum\limits_{n=1^{\infty}} \frac{1}{n(n+1)} = 1$ by using the fact that $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ to find that $S_N = 1 - \frac{1}{N+1}$, so it limits to 1.

Finally, we stated the fact that convergent series satisfy **linearity properties**.

Class 31: Tuesday, October 8. Throughout the class period, we worked on using techniques (including applying different theorems) to decide whether different sequences converge or diverge, and attempt to find their limit if they have one.

Class 30: Monday, October 7. We started class by stating the fact that if f is a continuous function and a sequence $\{a_n\}$ converges to a real number L, then the sequence $\{f(a_n)\}$ has limit $f(L)$.

From here, we defined what it means for sequences to be **bounded above**, **bounded below**, **bounded**, and **unbounded**. We stated the following **theorem**: If a sequence converges, then it must be bounded. However, we came up with several different examples where the converse does not hold.

We defined **increasing**, **decreasing**, **nonincreasing**, and **nondecreasing** sequences; we say that a sequence is **monotonic** if it has any of these properties.

Finally, we stated the following **theorem**, which essentially says that *monotonic bounded sequences converge*:

- If $\{a_n\}$ is eventually nondecreasing and $a_n \leq M$ for n large enough (in particular, the sequence is bounded above), then the sequences converges and its limit is at most M.
- If $\{a_n\}$ is eventually nonincreasing and $a_n \geq m$ for n large enough (in particular, the sequence is bounded above), then the sequences converges and its limit is at least m.

We also had a quiz on improper integrals.

Class 29: Friday, October 4. We continued to discuss sequences, giving several examples that all had the form of a **geometric sequence**, i.e., one of the form cr^n for some constants, c and r. We found the limit of a geometric sequence depending on these values.

Next, we stated the **limit laws** for sequences, which we will apply frequently.

After this, we turned to the question of whether the sequence $a_n = \frac{\sin(n)}{n}$ has a limit, and introduced the **squeeze theorem** for sequences to apply and conclude that its limit is 0!

After this, we introduced **factorials** of nonnegative integers, and used the squeeze theorem to prove that the sequence $b_n = \frac{5^n}{n!}$ limits to 0.

We pointed out that if $\lim_{x\to\infty} f(x)$ exists, then the sequences that "matches up" with this function a_n, i.e., $a_n = f(n)$, has the same limit. We saw an example of this.

Class 28: Thursday, October 3. We started class by finding appropriate comparison functions in some challenging examples, to show that a given improper integral converges or diverges.

Next, we introduced the notion of an **(infinite) sequence**. We discussed explicit and recursive formulas, the domain/index set of a sequence, and the graph of a sequence, through many examples.

We say that a sequence $\{a_n\}$ **converges** to limit L, and write

$$\lim_{n\ to\infty} a_n = L \text{ or } a_n \to L$$

if a_n gets *as close as desired* to L if we make n large enough. If no limit exists, we say that $\{a_n\}$ **diverges**, and if the terms a_n grow without bound, we say that the sequence **diverges to infinity**. We went through our examples, and studied the notion of a limit in detail.

Class 27: **Wednesday, October 2**. In teams, we worked on deciding whether a given improper integral converges or diverges, and also tried compute the value of an improper integral if it converges. However, we noticed that this is not possible if we simply conclude that an integral converges via a comparison. Many of the problems are taken from homework, so that we could get caught up after the midterm!

Class 26: **Tuesday, October 1**. Today, we finished our calculation from Friday, deciding when integrals of the form $\int_a^\infty \frac{dx}{x^p}$ converge or diverge, for p a constant. We found that for $a > 0$,

$$\int_a^\infty \frac{dx}{x^p} = \begin{cases} \text{diverges to } \infty & \text{if } p \leq 1 \\ \text{converges and equals } \frac{1}{1-p}a^{1-p} & \text{if } p > 1 \end{cases}$$

We reminded ourselves that an integral that is improper due having both $-\infty$ and ∞ as endpoints must be split up into two integrals. We found that $\int_{-\infty}^\infty e^{-x}\,dx$ diverges, since $\int_{-\infty}^0 e^{-x}\,dx$ diverges.

We then wrote $\int_0^\infty xe^{-x}\,dx$ as a limit, and found, using L'Hôpital's rule, that the integral converges and equals 1.

We considered the integral $\int_0^1 \frac{dx}{x}$, and noticed that we cannot just plug the endpoints into the antiderivative via the Fundamental Theorem, Part I; in fact, $x = 0$ is not even in the domain of the integrand! In fact, this integral is also improper for this reason, and we define

$$\int_0^1 \frac{dx}{x} = \lim_{R\to 0^+} \frac{dx}{x}.$$

We found that the limit approaches infinity, so that this improper integral diverges to infinity.

We then turned to the integral $\int_0^3 \frac{dx}{(x-1)^3}$, and notice that although the integrand is defined at the endpoints, it is not defined at the point $x = 1$ inside the interval of integration $[0, 3]$. We must break up the integral at the "bad point," and turn each resulting improper integral into a limit. We say that the original integral converges if both limits exist:

$$\int_0^3 \frac{dx}{x} = \int_0^1 \frac{dx}{(x-1)^3} + \int_1^3 \frac{dx}{(x-1)^3} = \left(\lim_{R\to 1^-} \int_0^R \frac{dx}{(x-1)^3}\right) + \lim_{S\to 1^+} \left(\int_S^3 \frac{dx}{(x-1)^3}\right)$$

assuming both limits exist (do they?).

We stated the convergence and divergence of integrals of the form $\int_0^a \frac{1}{x^p}\,dx$ for $a > 0$:

$$\int_0^a \frac{dx}{x^p} = \begin{cases} \text{converges and equals } \frac{1}{1-p}a^{1-p} & \text{if } p < 1 \\ \text{diverges to } \infty & \text{if } p \geq 1 \end{cases}$$

We then used this to show that $\int_1^\infty \frac{dx}{\sqrt{x^3+4}}$ converges, using the fact that $\frac{1}{\sqrt{x^3+4}} < \frac{1}{\sqrt{x^3}} = \frac{1}{x^{3/2}}$ and $\int_1^\infty \frac{dx}{x^{3/2}}$ converges by our conclusion above. To do so, we used the **Comparison Test** for improper integrals: Suppose that f and g are continuous functions for which $f(x) \geq g(x) \geq 0$ for $x \geq a$. Then

- If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ also converges.
- If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ also diverges.

A similar statement holds for improper integrals that are discontinuous at endpoints. We argued geometrically why this statement should hold.

Class 25: **Monday, September 30**. We worked on teams to solve several challenging problems on material that will be covered on Midterm 1. The first involved calculating the work to pump water over the top of a tank whose vertical cross-sections are trapezoids. Then we worked on finding antiderivatives of the following functions:

$$\frac{x^3-1}{x-1},\ \frac{x-x^3}{\sqrt{x}},\ x^3\sqrt{1+x^2},\ \sec^3 x,\ \frac{1}{\sqrt{\sqrt{x}+1}},\ x^3e^x,\ (3\sec x - \cos x)^2,\ \arctan x,$$

and $\frac{1}{x^2+4}$ via a trigonometric substitution.

Class 24: **Friday, September 27**. Our lecture today was motivated by the question of what the integral $\int_1^\infty \frac{1}{x^2}\,dx$ should mean. This is an example of an **improper integral**, which, in this case, is due to the fact that one endpoint is not a finite number. We drew a picture and decided it must "equal" the area under the graph of $y = \frac{1}{x^2}$ for $x \geq 1$, if this is finite. (Most students agreed that it *might* be finite!) We decided to define this integral as

$$\lim_{R\to\infty} \int_1^R \frac{1}{x^2}\,dx = \lim_{R\to\infty} \left(-\frac{1}{x}\right)\Big|_1^R = \lim_{R\to\infty} \left(1 - \frac{1}{R}\right) = 1.$$

Likewise, given a constant a and a function f that is integrable on any interval $[a, b]$ for $b > a$, we define the following improper integral in the following way:

$$\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx.$$

If this limit exists (and is a finite number), we say that the improper integral **converges** and equals this finite number. Otherwise, we say that the improper integral **diverges**.

We wrote out $\int_1^\infty \frac{1}{x}\,dx$ in terms of the appropriate limit, and found that the value approaches ∞. Hence this improper integral diverges to ∞. We noticed that $y = \frac{1}{x}$ lies above $y = \frac{1}{x^2}$ for $x > 1$, but they have a similar shape, so it is pretty subtle that one converges and the other diverges!

We described how to define improper integrals with $-\infty$ as an endpoint. For instance,

$$\int_{-\infty}^{0} \frac{dx}{x^2+1} = \lim_{R\to-\infty} \int_R^0 \frac{dx}{x^2+1}$$

which we found equals $-\frac{\pi}{2}$.

We also defined how to define an improper integral with endpoints $-\infty$ and ∞: We must "break up" the area at any point in the middle. For instance,

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \int_{-\infty}^{a} \frac{dx}{x^2+1} + \int_a^\infty \frac{dx}{x^2+1}$$

for any real number a; we chose $a = 0$. *Both* integrals must converge to say that the original integral converges! We motivated this by the intuition that $\int_{-\infty}^{\infty} x\,dx$ "should" diverge.

Finally, prompted by our earlier examples, we turned to the question of for which constants p, the integral $\int_a^\infty \frac{1}{x^p}\,dx$ converges, and for which p it diverges. (Here, a is any positive number.) We started by calculating the antiderivative of $\frac{1}{x^p}$ when $p \neq 1$, which is $\frac{1}{1-p}x^{1-p}+C$.

Class 23: **Thursday, September 26**. We noticed that some quadratic functions cannot be factored, and explained how the partial fractions decomposition method can be extended when there is a non-factorable quadratic term (or a power of such a term) in the denominator of a rational function. We used this fact to describe the general method of partial fractions decomposition: First factor the denominator, then decompose the integrand in terms of a sum of rational functions, with several unknown constants. Solve for the constants, and then find the antiderivative of each term.

We showed that if $a > 0$ is a constant, then

$$\int \frac{dx}{x^2+a} = \frac{1}{\sqrt{a}} \arctan\left(\frac{x}{\sqrt{a}}\right).$$

This formula will be useful when finding antiderivatives of rational functions!

We then worked in teams on problems that involved using the partial fractions decomposition method to find antiderivatives of the following functions:

$$\frac{x^3-x}{x^3+x^2-x-1},\ \frac{1}{(x+2)(x^2+4)},\ \frac{1}{(x+2)(x^2-4)},\ \frac{x^3+1}{x^2+1},\ \frac{dx}{2x^2-3},\ \frac{1}{x^2(x^2+25)^2}.$$

Class 22: **Wednesday, September 25**. We started class by noticing that although an antiderivative of a rational function (quotient of polynomials) such as $\int \frac{4x-14}{x^2-7x+12}\,dx$ can be found using a substitution, changing the integral slightly, say, to $\frac{2x-3}{x^2-7x+12}$ makes it so that this is not possible. We motivated the fact that our other methods of integration don't seem to be so helpful here, so we might want to try rewriting the integrand. Toward this, we noticed that the denominator of the integrand factors as $(x-3)(x-4)$, and claimed that not only can a function of the form $\frac{A}{x-3} + \frac{B}{x-4}$, for real numbers A and B, be written with a common denominator of $(x-3)(x-4)$, but the "opposite" also holds: a fraction with this denominator, and numerator a linear function, can always be written this way.

With this in mind, we set up the equation

$$\frac{2x-3}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4},$$

multiplied through by the denominator, and then plugged in values $x = 3$ and $x = 4$ to solve for A and B. We found that $A = -3$ and $B = 5$, so that

$$\int \frac{4x-14}{x^2-7x+12}\,dx = -\int \frac{3}{x-3}\,dx + \int \frac{5}{x-4}\,dx = -3\ln|x-3| + 5\ln|x-4| + C.$$

We pointed out that this general method always works to find the antideritative of a rational function (fraction of polynomials), assuming 1) The denominator factors into *distinct* linear terms, and 2) the degree (highest power of x appearing) in the numerator is less than the degree of the denominator. This method is called **partial fractions decomposition**.

Using this idea, we found the antiderivative of $\frac{x^2+2x-44}{(x+3)(x+5)(3x-2)}$, after writing it as $\frac{A}{x+3} + \frac{B}{x+5} + \frac{C}{3x-2}$ for some real numbers A, B, and C, and then integrating each piece. Notice that the antiderivative of $\frac{1}{3x-2}$ is *not* $\ln|3x-2|$!

We turned to the case that the degree of a rational function's numerator is not less than that of its denominator, writing

$$\frac{1}{x^2-1} = \frac{x^2-1}{x^2-1} + \frac{1}{x^2-1} = 1 + \frac{1}{x^2-1}$$

and then integrating each term. The second term again requires partial fractions decomposition!

We turned to finding the antiderivative of $\frac{x^3-x}{x^3+x^2-x-1}$, first using the method above to reduce the problem to one where the degree of the numerator is smaller than 3, and then turned to factoring the denominator. We immediately noticed that $x = 1$ and $x = -1$ are roots of this cubic polynomial, so that $(x-1)(x+1)$ is a factor of the denominator. Then we used polynomial long division to see that the remaining factor is again $x-1$, so that

$$x^3 + x^2 - x - 1 = (x-1)(x+1)^2.$$

Unfortunately, $\frac{x^3-x}{x^3+x^2-x-1} = \frac{x^3-x}{(x-1)(x+1)^2}$ cannot necessarily be written as $\frac{A}{x-1} + \frac{B}{x+1}$ for real numbers A and B, nor necessarily as $\frac{A}{x-1} + \frac{B}{(x+1)^2}$. However, we claim that it *can* be written as

$$\frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

for some real numbers A, B, and C. Tomorrow we will see the general pattern, and then work on some problems related to this!

Class 21: **Tuesday, September 24**. We worked in teams on problems involving trigonometric substitution, seeking the antiderivatives of:

$$\frac{1}{\sqrt{x^2+9}},\ \frac{1}{x^2\sqrt{4x^2-36}},\ \frac{1}{x^2+a},\ \frac{1}{x^2\sqrt{x^2-2}}$$

where a is a constant. Remember that we had to be *very* careful with restricting the domain of the new variable!

Class 20: **Monday, September 23**. We started class by remembering our problem of finding $\sqrt{1-x^2}\,dx$ from last time, and the basic method using the substitution $x = \sin\theta$, for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

We noticed that by the same type of argument, integrals involving a term of the form $\sqrt{a^2-x^2}$ can often be found by using the substitution $x = a\sin\theta$, for $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

We then found $\int \frac{dx}{\sqrt{4-x^2}}$ using $x = 2\sin\theta$. We noticed that $x = \sqrt{3}\sin\theta$ should work for $\int \frac{x}{\sqrt{3-x^2}}\,dx$, but that it might be easier to use the simple substitution $u = 3-x^2$. However, we likely need this trig substitution for, say, $\int \frac{\sqrt{3-x^2}}{x}\,dx$.

We went through, in detail, the computation of the antiderivative

$$\int \frac{x^2}{(9-x^2)^{3/2}}\,dx = \int \frac{x^2}{(\sqrt{9-x^2})^2}\,dx$$

which involved restricting the domain of θ to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, calculating all values to substitute, rewriting the integrand using a trigonometric identity, finding the antiderivative of each term, using a right triangle to solve for terms of antiderivative in terms of x, and making sure that we can apply inverse trig functions on our domain.

Finally, we posed the problem of finding $\int \frac{dx}{\sqrt{x^2+25}}$, and after trying to substitute $x = 5\cos\theta$ and failing, we investigated whether $x = 5\tan\theta$ or $x = 5\sec\theta$ might work. In fact, a trigonometric identity allows us to substitute the entire integrand for $x = 5\tan\theta$, and we proceeded in an analogous way (encountering different questions at each stage!) to find the antiderivative in terms of θ. Notice that we needed to restrict the values of θ carefully, again! Before class tomorrow, try writing the antiderivative in terms of x.

Class 19: **Friday, September 20**. We began class by giving a hint as to how to find $\int \sec^3 x\,dx$ or $\int \csc^3 x\,dx$.

Next, motivated by the fact that definite integrals of the form $\int_{-a}^{a} \sqrt{a^2-x^2}\,dx$ have appeared in many of our problems on applications of calculus, but our only way to find these are to realize the value in terms of the area of a circle, we posed the question of how to find antiderivatives involving

$$\sqrt{a^2-x^2}, \sqrt{a^2+x^2}, \text{ or } \sqrt{x^2-a^2}.$$

We started attacking this question by studying

$$\int \sqrt{1-x^2}\,dx.$$

Motivated by the fact that $\sin^2\theta+\cos^2\theta=1$, we investigated whether substituting $x=\sin\theta$ would make sense. We concluded that since the domain of the integrand $y=\sqrt{1-x^2}$ is $[-1,1]$, so that if we restrict θ to take on values in $[-\pi/2,\pi/2]$, the substitution $x=\cos\theta$ makes sense.

We started the substitution, noticing that if $x=\sin\theta$, then $dx=\cos\theta\,d\theta$, and

$$\int \sqrt{1-x^2}\,dx=\int \sqrt{1-\sin^2\theta}\cos\theta\,d\theta=\int \sqrt{\cos^2\theta}\cos\theta\,d\theta.$$

Now, since $\cos\theta\geq 0$ for $\theta\in[-\pi/2,\pi/2]$, we have that $\sqrt{\cos^2\theta}=\cos\theta$ for these values of θ, so that this integral equals $\int\cos^2\theta\,d\theta$. We remembered that we can find this antiderivative using integration by parts twice, obtaining $\frac{1}{2}\theta+\sin\theta\cos\theta+C$. Then we used labeled the legs of a right triangle with angle θ so that $\sin\theta=x=x/1$; we chose the side opposite to the angle to have length x, so that the adjacent leg has length $\sqrt{1-x^2}$.

We noticed that for our values of θ, $x=\sin\theta$ is exactly the same as saying $\theta=\arcsin x$. Using this triangle to solve for the remaining values in our antiderivative, we concluded that

$$\int \sqrt{1-x^2}\,dx=\frac{1}{2}\theta+\sin\theta\cos\theta+C=\frac{1}{2}\arcsin x+x\sqrt{1-x^2}+C.$$

Class 18: **Thursday, September 19**. We began class by finding the antiderivatives of $\tan x$ and $\sec x$ by rewriting the integrand. We reminded ourselves how to derive trigonometric identities $1+\tan^2 x=\sec^2 x$ and $1+\cot^2 x=\csc^2 x$ from the identity $\sin^2 x+\cos^2 x=1$

Then in teams, we found $\int\tan^2 x\,dx$, $\int\tan^2 x\sec^3 x\,dx$, and $\int\cos^{498} x\sin^3 x\,dx$. After this, we worked on either the homework problem 6.5, #21 on work, or the problems assigned as homework today.

Class 17: **Wednesday, September 18**. First, we fully completed two problems from yesterday involving the calculus of parametic equations. In particular, we calculated the slope of the tangent line to a circle at several points, and the area of a circle, using this method.

Next, we approached the question of how to find an antiderivative of the form

$$\int \sin^m x,\cos^n x\,dx$$

where m and n are nonnegative integers, so that when $n=0$ or $m=0$, we are considering $\int\sin^m x\,dx$ or $\int\cos^n x\,dx$, respectively.

We first noticed that $\int \cos^3 x\,dx$ can be computing by writing it as

$$\int \sin^2 x \sin x\,dx = \int (1-\cos^2 x)\sin x\,dx$$

and using the substitution $u = \cos x$. We saw that a similar method can be done to find

$$\int sin^5 x \cos^6 x\,dx$$

after writing $\sin^5 x = (1-\cos^2 x)^2 \sin x$, and using $u = \sin x$.

We noticed that via this general method of gathering some "$\sin^2 x$" or "$\cos^2 x$" terms, and then rewriting the integrand using the trigonometric identity $\sin^2 x + \cos^2 = 1$, one can always come to a lone "$\sin x$" or "$\cos x$" term when **at least one of the powers m or n is odd**. In this case, we can use a substitution $u = \cos x$ or $u = \sin x$, respectively, and then the power rule, to find the antiderivative.

Next, we turned to the case when neither power is odd, first approaching $\int \sin^m x\,dx$ or $\in \cos^n x\,dx$. The integrand $\int \sin^2 x\,dx$ actually appeared in our parametric example earlier today in computing the area of a circle; we had solved it by using the double angle formula $\sin^2 x = \frac{1}{2}(1-\cos(2x))$. We can do the same for $\int \cos^2 x\,dx$ using $\cos^2 x = \frac{1}{2}(1+\cos(2x))$.

We then noticed that $\int \sin^4 x\,dx$ can be found by first writing the integrand as $(\sin^2 x)^2$, using the double angle formula, expanding, and applying it again (try to re-do this yourself!). Likewise, this method can be used iteratively to compute the integral of an **even power of sine or cosine**.

Finally, we saw that if there is a **positive even power of sine *and* cosine**, we can use the identity $\sin^2 x + \cos^2 x = 1$ to obtain a sum of powers of sine or cosine, with constant coefficients, such as:

$$\int \sin^4 x \cos^2 x\,dx = \int \sin^4 (1-\sin^2 x)\,dx = \int \sin^4 x - \int \sin^6 x\,dx.$$

and each of the resulting antiderivatives can be found via the previous method.

We finished by posing the question of how to find $\int \tan x\,dx$ "from scratch." Think about it!

Class 16: Tuesday, September 17. We started class by finding parameterizations for the line going through a point (a, b) with slope m:

$$x = a + rt, y = a + st$$

where t is any real number, and $m = s/r$.

We then found, using the chain rule, an equation for the slope of the tangent line to a parametric curve $c(t) = (x(t), y(t))$ when $x(t)$ and $y(t)$ are differentiable and $x'(t)$ is continuous and nonzero:

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{y'(t)}{x'(t)}.$$

We also used the substitution method to compute the area under the graph of a parametric curve $c(t) = (x(t), y(t))$ for $a \le x \le b$, if $x(t_0) = a$ and $y(t_1) = b$:

$$A = \int_{t_0}^{t_1} y(t)x'(t)\,dt.$$

Then, in groups, we applied these concept. We first "eliminated the parameter" to write parametric equations as functions y in terms of x, when possible, in 11.1: 8, 12, and 14. Then we wrote a parameterization for a circle centered at $(1, -5)$ with radius 4, tracing out the curve both clockwise, and counterclockwise. After this, we aimed to compute the slope of the tangent line to this circle at certain points, and then compute the area of a certain circle parametrically. However, most groups did not finish these last parts, so we'll talk about them tomorrow.

Class 15: **Monday, September 16**. We finished our example from last time of calculating the work required to construct a concrete pillar. Next, we calculated the work required to pump water from a spherical tank out of a spout above it. Each required us to break the problem up into small pieces so that the *distance* of moving one piece is approximately constant; then we could use an integral to calculate total work.

We then turned to the study of **parametric equations**. Writing a curve parametrically allows us to model curves in which y is not necessarily a function of x, and to model movement through time. A parametric equation is given by $x = x(t), y = y(t)$ for t a real number, possibly in a restricted domain. We sometimes use $c(t) = (x(t), y(t))$ to denote the *parametric curve* given by the parametric equations.

We studied several ways of writing $y = x^2$ parametrically, first as (t, t^2) for all real numbers t, and compared what changes when we restrict to $t \in [-1, 4]$. We then described the curves (t^3, t^6), (t^2, t^4), and $(\sin t, \sin^2 t)$ for t any real number.

Finally, we modeled the unit circle parametrically in two ways, and then modified the parameterization to trace out the circle of radius 2 centered at the origin.

Class 14: **Friday, September 13**. We started class today by solving the problem posed yesterday, which involved identifying the "top" bound of integration in 6.4, #28.

Next, we recalled how to calculate the energy, **work**, required when a constant force is applied to an object to move it, via the equation Work = Force × Distance. We discussed the units of force and energy, and did some simple examples to illustrate them.

Next, we determined a formula for the work required to move an object along the x-axis from $x = a$ to $x = b$, when the force applied has magnitude $F(x)$:

$$\int_a^b F(x)\,dx.$$

We did a few examples of applying **Hooke's law** for springs. Next, we set up a problem in computing the work required to construct a pillar with square base. We'll finish this problem next time!

Class 13: **Thursday, September 12**. After a quiz on integration by parts and volumes of revolution, we did problems involving the method of cylindrical shells. We did problems inspired by 6.4: 14, 20, 28, and 32.

Class 12: **Wednesday, September 11**. Today we discussed another method of integration the **method of cylindrical shells**. Essentially, we divide a solid of revolution into pieces that are approximated by the "shells" of cylinders. By "unrolling" the shell, we determined that if the shell's height is perpendicular to the x-axis, then the volume of each shell is approximately

$$(2\pi \cdot \text{radius})\,(\Delta x)(\text{height}),$$

the product of length, width, and height. If the shell's height is perpendicular to the y-axis, "Δx" is replaced with "Δy" in the expression. The total volume of the solid of revolution is then

$$2\pi \int_a^b (\text{radius})(\text{height})\,dx$$

for appropriate bounds of integration $a \leq x \leq b$, or an analogous formula if x is replaced with y.

We did an example, computing a volume of revolution using the disc method, and then the method of cylindrical shells. We noticed pros and cons of each method; sometimes it is difficult to solve for a value necessary in one integrand, and sometimes one antiderivative is much harder to find. In the two methods, the variable of integration changes. We also noticed that in the disc and washer method, the radius of the disc is perpendicular to the axis of rotation, and in the cylindrical shell method, the shell height is parallel to the axis of rotation.

We started setting up the integral, using our new method, giving the volume of revolution of the region between $x = y(4-y)$ and $x = (y-2)^2$, after being revolved about the x-axis. Try to finish this before class tomorrow!

Class 11: **Tuesday, September 10**. We worked in teams on problems involving the computation of volumes of revolution. These were (slight modifications of) the following problems from 6.3: 14, 19, 27, 30, 31, 58, 59. The last two are also assigned as homework!

Class 10: **Monday, September 9**. Today, we described **solids of revolution** about a horizontal or vertical axis.

If f is a continuous function on $[a, b]$, and $f(x) \geq 0$ on this interval, consider the area below the graph of f for $a \leq x \leq b$. If this region is rotated about the x-axis, consider the solid swept out. The cross section at any x-value will be a circle of some radius $R = f(x)$, so the area of the cross section is $\pi R^2 = \pi[f(x)]^2$, and the volume of the entire solid is

$$\int_a^b \pi R^2\, dx = \pi \int_a^b (f(x))^2\, dx.$$

This formula (or its derivation) is often called the **dis method** for computing volumes of revolution.

We did an example, computing the volume of the solid obtained after rotating the region under the graph of $y = x^3$ for $0 \leq x \leq 1$ about the x-axis. We then considered the region between $y = x^3$ and $y = x^2$, between their points of intersection. To compute the volume of the solid obtained when this region is revolved about the x-axis, we derived the **washer method** for computing volumes of revolution, where a cross-section is the shape of a "washer," a disc of radius R_{outer} with a disc of radius R_{inner} removed:

$$\pi \int_a^b (R_{\text{outer}}^2 - R_{\text{inner}}^2)\, dx.$$

We then used this formula to find the volume of revolution in our example.

Next, we found the volume of revolution of a solid obtained after revolution about a horizontal line that is not the x-axis. Finally, we compute a volume of revolution of a solid obtained by revolving about a vertical axis.

Class 9: **Friday, September 6**. Today's class started with a brief lecture motivating and defining the **average value** of a function $f(x)$ on a closed interval of input values $[a, b]$:

$$\int_a^b f(x)\, dx.$$

In (new) teams, we completed the following:

A. Find the average value of the function $y = x^2$ on the interval $0 \leq x \leq 10$. Is it more or less than 50? Can we describe why this should be the case based on the graph?

B. Looking at the graphs, should the average value of $y = \sin x$ on $[0, \pi]$ be greater than, less than, or equal to the average value of the semicircle $\sqrt{1 - x^2}$ on $[-1, 1]$?

Next, we worked through some problems, finding volumes of solid figures using our two-step process from last time:

1. Find the volume of a cone with height 7 and radius 12.
2. Find the volume of the solid with base the unit circle $x^2 + y^2 = 1$, and whose vertical cross sections are equilateral triangles.
3. Problem 6.2, #19 on *frustrums*

Most groups did not get to the last problem, and it is assigned as homework from the textbook.

Class 8: Thursday, September 5. We began class today by asking what quantities integrals can represent, and then recalled our first motivation for an integral, total distance traveled in terms of velocity.

We next turned to *volumes*. We motivated, and then demonstrated together that given a solid body extending from $y = a$ to $y = b$ in which the cross-sectional area at height y equals $A(y)$ has volume

$$\text{Total volume} = \int_a^b A(y)\,dy.$$

We noticed that to apply this formula, we should always first **(1)** Find a formula for $A(y)$, and then **(2)** Compute the volume via the definite integral formula.

We found the volume of a certain pyramid using this formula, where we used a "similar triangles" technique to compute the cross-sectional area.

Next, we found the volume of an arbitrary sphere of radius R using calculus, $V = \frac{4}{3}\pi R^3$!!

Class 7: Wednesday, September 4. Today, we deduced a formula for the area between the graphs of two functions, $f(x)$ and $g(x)$, on the interval $a \leq x \leq b$, assuming that f lies above g on the interval (meaning that $f(x) \geq g(x)$ for all x in the interval):

$$\int_a^b \left(f(x) - g(x)\right)\,dx.$$

Throughout the class period, we applied this formula to compute areas, but came across several interesting subtleties in which we had to be careful.

First, we computed a certain region between $y = \sin\theta$ and $y = \cos\theta$: the region in the interval $[0, 2\pi]$ in which $\sin\theta \geq \cos\theta$. We had to determine what θ-values bound this region, and we found that they are $\theta = \pi/4$ and $5\pi/4$, so that the area equals

$$\int_{\pi/4}^{5\pi/4} (\sin\theta - \cos\theta)\,d\theta = 2\sqrt{2}$$

(this final answer was computed after finding an antiderivative, and applying the Fundamental Theorem of Calculus, Part I).

Next, we considered the region between the parabola $f(x) = x^2 + 5$ and the line $g(x) = -4x + 17$ for $-3 \leq x \leq 5$. Here, sometime f lies above g, and sometimes g lies above x, and we sketched graphs of the functions, and solved for intersection points, to determine where each of these occur. We found that g lies above f for $-3 \leq x \leq 2$ and vice versa for $2 \leq 5$, so the area is:

$$\int_{-3}^{2} (-4x + 17) - (x^2 + 5)\,dx + \int_2^5 (x^2 + 5) - (-4x + 17)\,dx.$$

After this, we considered a region bounded between *three* curves: $y = \frac{1}{2x}, y = x$, and $y = x^2$. After sketching the graph and finding all pairwise intersection points, we determined that there are two different regions bounded between all curves, and described each in terms of integrals (sometimes differences of them!).

Finally, if $x = g(y)$, then we noticed that the signed area between g and the y-axis can be computed as $\int_c^d g(y)\,dy$, assuming that the region is bounded between $y = c$ and $y = d$.

Class 6: **Tuesday, September 3**. We started class with a quiz on substitution, and then we worked on groups finding antiderivatives, by applying integration by parts (and possible substitution as well). In the middle of class, we introduced the integration by parts formula for definite integrals:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

and we worked on some problems that involved applying this formula. (If you did not finish these, make sure to do them for homework!)

Check out the *Quiz Correction Guidelines* posted on the course website!

Class 5: **Friday, August 30**. We started class by finding two antiderivatives using the substitution method, that provided extra (new) challenges: $\int r\sqrt{5-\sqrt{4-r^2}}\,dr$ and $\int \frac{\sin^4\theta}{\sec^3\theta}\,d\theta$. The first required two substitutions, one of which involving solving for the original variable in terms of the substituted one, and the second required rewriting the integrand in terms of $\sin\theta$ and $\cos\theta$, and then rewriting it *again* using the identity $\sin^2\theta + \cos^2\theta = 1$.

After this, we derived by hand the method of **integration by parts** for finding antiderivatives, which was motivated by our desire to find the antiderivate of a product of two functions:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

If $u = f(x)$ and $v = g(x)$, this translates to

$$\int u\,dv = uv - \int v\,du.$$

In the same vein that the substitution method for antiderivatives is a "reverse" to the chain rule for derivatives, integration by parts reverses, in a slightly more subtle sense, the product rule for derivatives.

We saw from the examples $\int x\cos x\,dx$ and $\int x\ln x\,dx$ that the choice of which function is u, and which becomes part of dv matters: We noticed that we should keep in mind that the choice for dv must be a function for which we can compute the antiderivative, and we need that $\int v\,du$ is somehow accessible as well.

Finally, we found $\int x^2e^x\,dx$ via two iterative applications of integration by parts, and then $\int e^x\sin x\,dx$ by applying the method twice and then *solving* for the antiderivative, which ended up appearing twice in our expression.

Next time we will find $\int \ln x\,dx$ via integration by parts!

Class 4: Thursday, August 29. Today, we first finished working on some substitution problems from last time. From here, we introduced and motivated the **change of variable formula for definite integrals**:

$$\int_a^b f(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du.$$

On the other hand, if we can find an antiderivative using the substitution method, we can, of course, use that answer to compute the definite integral with the same integrand using the Fundamental Theorem of Calculus, Part II. We did a few examples, including one in terms of areas, and then worked on teams on some more challenging problems.

Be ready for a quiz on the substitution method either tomorrow, or early next week!

Class 3: Wednesday, August 28. We began class by recalling how to define and compute a definite integral using the limit definition, and why this measures the (signed) area between the graph and the x-axis.

The Fundamental Theorem of Calculus, Part I makes this calculation much easier, at least if we can find an antiderivative! (Our example of e^{x^2} is one we'll come back to, in which we cannot currently find one!) Our goal for the immediate future in this course is to develop tools to use that help us find antiderivatives.

We presented the **substitution method** for finding antiderivatives: If F is an antiderivative for f, then

$$\int f(u(x))u'(x)\,dx = F(u(x)) + C.$$

We showed why this holds, and argued that this could also be called the "reverse chain rule."

We also presented the notion of differentials, and gave the change of variables formula:

$$\int f(u(x))u'(x)\,dx = f(u)\,du.$$

Next, we worked in teams on several (increasingly difficult) antiderivatives that require application of the substitution method.

Class 2: Tuesday, August 27. Today we began class by working in teams to find the derivatives of $f(x) = x^3$ (first, its value at $x = 2$) and $g(x) = |x|$ using the definition of the derivative.

After this, we recalled the definition of an **antiderivative** and an **indefinite integral**, and compared this to the notion of a **(definite) integral** that measures the (signed) area between the graph of a function and the x-axis. We used the geometric interpretation of definite integrals to compute them without using antiderivatives, and we ended by recalling the deep connection between definite integrals and antiderivatives via the **Fundamental Theorem of Calculus, Part I**.

Tomorrow we will work in teams in learning, and applying, our first rule for computing antiderivatives!

Class 1: Monday, August 26. We began the course by first going over the syllabus and the expectations for the semester. Next, we gave a lightning-fast summary of Calculus I material, emphasizing what is most important to know, and take away, from this course.

In particular, we discussed how the concept of a **limit** differentiates calculus from all math that comes “before” calculus, and tied the limit definition of the derivative to the geometric notion of (instantaneous) slope. We gave several examples, and noticed that the “derivative rules” do *not* apply to all functions for which we have equations.

Next time, we will tie Calculus I material to integrals, and start with techniques of integration. Make sure to send my your first assignment!