Daily Update

Math 145, Fall 2017

Lecture 64: Thursday, December 7. Today, we worked through a wide variety of problems covering material throughout the semester.

Good luck studying for the final exam! It's been a great class.

Lecture 63: Wednesday, December 6. We reviewed the statement of integration by parts, and did several more examples in order to identify good choices for u and dv, and see that sometimes this process can be used twice or combined with substitution to solve a problem.

Next, we stated **integration by parts for definite integrals**

$$\int_a^b u\,dv = (uv)|_a^b - \int_a^b v\,du = u(b)v(b) - u(a)v(b) - \int_a^b v\,du.$$

We applied this to solve several problems.

After this, we looked over a bunch of integrals, and decided on what method would be advantageous.

Lecture 62: Monday, December 4. We introduced our last topic today, the anti-differentiation method **integration by parts**. Motivated by the fact that $\int x^2 \cos x\,dx$ is mysterious, while it is clear by the product rule that

$$\int (x^2 \cos x - 2x \sin x)\,dx = x^2 \sin x + C,$$

we derived the general formula

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx,$$

or, if $u = f(x)$ and $v = g(x)$,

$$\int u\,dv = uv - \int v\,du.$$

This helps us find $\int f(x)g'(x)\,dx = \int u\,dv$ if it is possible to find the new antiderivative appearing on the right-hand side of these equations.

We found $\int x \cos x\,dx$ and $\int xe^x\,dx$ using integration by parts, We noticed that if $u = x$ and we are able to find the antiderivative of dv, we can always apply this method to find an antiderivative.

We saw that this method can be fruitful even when an integrand does not look like a product. For example, we found that

$$\int \ln x = x \ln x - x + C,$$

using $u = \ln x$ and $dv = dx$. (We couldn't use $dv = \ln x$ since this would be the same as solving the original problem!)

We found that for $\int e^x \sin x \, dx$, we could apply integration by parts *twice* to find the antiderivative. For $\int \sin x \cos x \, dx$, we could apply the method and then subtract the new integral the left-hand side to solve, but the substitution method also works!

Finally, we considered $\int \sin^2 x \, dx$, and found that applying integration by parts twice did not help! However, we can use the identity $\sin^2 x + \cos^2 x = 1$; we'll complete this next time!

Lecture 61: Friday, December 1. Today, we reviewed the methods we've developed to find antiderivatives and definite integrals by working through several problems. In each case, we needed to identify what method to use.

We also discussed proposals for our final projects, in teams. We finished class with our final quiz of the term.

Lecture 60: Thursday, November 30. We started class today by manipulating an integrand to make it possible to do a substitution, so that eventually we can find an antiderivative in terms of inverse trigonometric functions, building on yesterday's lecture. We spent the second part of class working on increasingly challenging problems related to this.

Lecture 59: Wednesday, November 29. Today we used calculus and induction to show that for any integer $n \geq 1$,

$$e^x \geq 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!},$$

which actually gave us a nice estimate for e, which alludes to some Calculus II material.

Next, we started looking at some less obvious substitutions, involving the natural logarithm and inverse trigonometric functions.

Lecture 58: Tuesday, November 28. We worked in teams on three proofs using mathematical induction.

Lecture 57: Monday, November 27. Today we discussed the **Principle of Mathematical Induction**: Suppose that $S(n)$ is a statement about integers $n \geq a$ for some integer a. If

- (Base case) $S(a)$ is true, and
- Assuming $S(k)$ is true for some $k \geq a$, then $S(k+1)$ is also true,

then $S(n)$ is true for all integers $n \geq a$.

We gave several examples of statements $S(n)$. Then we proved:

- For all integers $n \geq 1$,
$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

- For all $n \geq 1$
$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

The homework problems today will be from Childs' book, which appears in Blackboard.

Lecture 56: Monday, November 20. We stated and applied the extension of the **substitution rule to definite integrals**:

$$\int_a^b f(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du.$$

Then we completed several challenging problems in teams on that apply the substitution method for antiderivatives.

Lecture 55: Friday, November 17. We investigated a series of antiderivatives, and noticed that they all satisfy the same property, where we can "reverse the chain rule." For example, $\int 2x\sin(x^2)\,dx$ equals $-\cos(x^2) + C$, since we know the antiderivative of $\sin x$ is $-\cos x$, and the derivative of x^2 appears. Formalizing this, we stated the **substitution rule** for antiderivatives:

$$\int f(u(x))u'(x)\,dx = F(u(x)) + C,$$

if F is an antiderivative of f. We verified that this holds using the chain rule.

We did many applications of the substitution rule, including cases where we have to be a bit creative in the choice of substitution.

Lecture 54: Thursday, November 16. We started class by answering homework questions on FTC II (§5.5), and then worked on some creative/challenging problems related to this theorem in teams.

Lecture 53: Wednesday, November 15. Today we first worked in detail with the area functions $A_a(x) = \int_a^x f(t)\,dt$ associated to a function $f(x)$ that is continuous on an open interval containing $x = a$. In fact, FTC II tells us that as a varies, these are all the antiderivatives of f on this interval!

We used functions of $f(x)$ to graph $A_a(x)$ in two cases, and saw that as a changes, we indeed obtain shifts up and down of one another. In one case, $f(x)$ was a piecewise function, and its antiderivative on each piece had different choices of constant so that $A_a(x)$ was continuous.

Next, we discussed **net change** of a function over a time interval, and wrote it as the a definite integral of the rate of change function with respect to time. In particular, if $r(t)$ is the rate of change of $s(t)$, then the net change on the interval $[t_1, t_2]$ is

$$s(t_2) - s(t_1) = \int_{t_1}^{t^2} s'(t)\,dt = \int_{t_1}^{t^2} r(t)\,dt.$$

Our main examples involved the rate of flow of water, and velocity. If $s(t)$ is position with respect to time, and $v(t)$ denotes velocity at time t, then we deduced that the **total displacement** on $[t_1, t_2]$ equals

$$s(t_2) - s(t_1) = \int_{t_1}^{t^2} v(t)\,dt.$$

On the other hand, **total distance traveled** equals

$$\int_{t_1}^{t^2} |v(t)|\,dt$$

We saw each graphically.

Lecture 52: Tuesday, November 14. We started class by reviewing the FTC II, sketching the area function, and reminding ourselves why any two area functions differ by a constant. The big upshot of the FTC II is that an antiderivative of any continuous function exists!

We did several problems involving finding derivatives of area-like functions, several of which needed the chain rule, or to rewrite the integral so that the endpoints satisfy the FTC II.

Next, we proved the FTC II in the case that the original function is increasing.

Finally, we started working on a problem involving finding the equation and graph of an antiderivative of a piecewise function.

Lecture 51: Monday, November 13. Today, we reviewed the statement of the Fundamental Theorem of Calculus, Part I (FTC I), and noticed the defect might be that there is no antiderivative of a given function.

Next, we pointed out how the families of antiderivatives $\int f(x)\,dx$ and $\int f(t)\,dt$, and then the numbers $\int_a^b f(x)\,dx$ and $\int_a^b f(t)\,dt$, compare.

We defined a function $A_a(x)$ associated to a function $f(x)$ on an open interval I, and a real number a in I, as the signed area under the graph from a to x,

$$A(x) = \int_a^x f(t)\,dt.$$

It was immediately clear that $A_a(a) = \int_a^a f(t)\,dt = 0$, and if $x < a$, then $A_a(x) = \int_a^x f(t)\,dt = -\int_{-x}^a f(t)\,dt$.

We calculated $A_0(x)$ and $A_{-3}(x)$ corresponding to $f(x) = x^2$, and saw that they differ precisely by the constant $\int_{-1}^0 f(t)\,dt$. We also found a piecewise formula for the function $A_0(x)$ for $f(x) = |x|$.

Motivated by our work so far, we stated the **Fundamental Theorem of Calculus, Part II** (FTC II): If f is a continuous function on an open interval I, and a is a real number in I, then the area function

$$A_a(x) = \int_a^x f(t)\,dt$$

is an antiderivative for f on I. In other words, $A_a'(x) = f(x)$, or

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

We noticed that this worked for our equations for $A_0(x)$ and $A_{-3}(x)$ corresponding to $y = x^2$, and then saw a few quick consequences.

Lecture 50: Friday, November 10. Today, we worked on problems related to FTC I in teams.

Lecture 49: Thursday, November 9. In class today, we firs practiced finding all solutions to the differential equation from Tuesday's lecture, $f''(x) = x^5 - 3x$. Then we solved an *initial value problem* for this equation, where values for $f(0)$ and $f'(0)$ were given.

Next, motivated by the fact that Riemann sums, so also definite integrals, can be hard to find, we stated the **Fundamental Theorem of Calculus, Part I** (FTC I): If f is continuous on $[a, b]$, and F is an antiderivative of f on $[a, b]$ (i.e., $F'(x) = f(x)$ for $x \in [a, b]$), then

$$\int_a^b f(x)\,dx = F(a) - F(b).$$

We carefully studied the hypotheses of this theorem, and also noticed that the right-hand side of this equation is independent of the choice of antiderivative.

We applied the theorem in several examples, and pointed out that it agrees with our computation of $\int_0^a x^2\,dx$ from Tuesday's lecture. Next, we noticed that the theorem doesn't appear to be so helpful unless we know an antiderivative of f; e.g.,

$$\int_0^1 e^{x^2}\,dx$$

is mysterious.

Finally, we proved FTC I, relating Riemann sums to an antiderivative!

Lecture 48: Wednesday, November 8. We worked through several challenging problems together on Midterm 2 material. In particular, we applied L'Hôpital's rule creatively, set up and solved an applied optimization problem, and found a definite integral with variable endpoints using Riemann sums.

Lecture 47: Tuesday, November 7. We started class by defining an **antiderivative** of a function on an open interval. Next, we found antiderivatives of several functions, and then encountered some that we could not find. We noticed that shifting a function up or down does not change its derivative at any point, and verified that if $F(x)$ is an antiderivative of $f(x)$ on an open interval, then so is $F(x)+C$ for any real number C. Amazingly, we stated and proved the converse of this observation: *all* antiderivatives on the open interval must be of the form $y = F(x) + C$. This required the use of the Mean value theorem!

Next, we defined the *indefinite integral* of a function f on an open interval as

$$\int f(x)\,dx = F(x) + C,$$

where F is an antiderivative of f on the interval and C signals a constant. By the theorem just proved, nomatter the choice of F, as the constant varies, we obtain all antiderivatives of f on the interval.

We found some indefinite integrals, and derived the **power rule** for antiderivatives:

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \text{ if } n \neq 1.$$

When $n = 1$, we noticed that the formula doesn't work, but that

$$intx^{-1} \, dx = \int \frac{1}{x} \, dx = \ln(x) + C,$$

but this works just when we only consider $x > 0$. In fact, the domain of $y = \frac{1}{x}$ consists of *two* open intervals, $(-\infty, 0)$ and $(0, \infty)$. Graphically, and then equationally, we showed that

$$\int \frac{1}{x} \, dx = \begin{cases} \ln(x) + C & \text{if } x > 0 \\ \ln(-x) + C & \text{if } x < 0 \end{cases} = \ln|x| + C.$$

We found some antiderivatives of more complicated functions, and noticed that we were implicitly using **linearity of the indefinite integral**, which follows from linearity of the derivatives:

$$\int (f(x) \pm g(x)) \, dx = \int f(x) \, dx \pm \int g(x) \, dx, \quad \text{and} \quad \int kf(x) \, dx = k \int f(x) \, dx$$

on an open interval, for k a constant.

We noticed that we could *not* immediately find antiderivatives of functions that consisted of products, quotients, and compositions of other functions. In general, we'll need to find some kind of "anti-product rule," "anti-quotient rule," and "anti-chain rule."

On the other hand, when we have a composition of functions $f(kx)$, where k is a constant and f is a function for which we can find an antiderivative F, we found through the specific examples of $f(x) = \cos(x)$, $\sin(x)$, and e^x, that in general

$$\frac{1}{k} F(kx)$$

is an antiderivative of $f(kx)$.

Finally, we gave the problem of finding one/all solution(s) of the differential equation

$$f''(x) = x^5 - 3x.$$

Lecture 46: Monday, November 6. Today, we returned to curve sketching, and worked in detail, especially focusing on asymptotes. We sketched the curve $f(x) = \frac{1}{x^2+x-6}$ using derivatives and limiting information, which included two vertical asymptotes and a horizontal asympote. Then we found the *slant asymptote* of the function $g(x) = \frac{x^2}{x-1}$, which is $y = x+1$ and sketched its graph.

Finally, to lead into tomorrow's lecture, we defined an **antiderivative** $F(x)$ of a function $f(x)$ as one for which

$$F'(x) = f(x).$$

We will revise this definition slightly tomorrow, and go into more detail. Soon we will see on of the most magical things in calculus: how the notion of an antiderivative relates to the definite integral!

Lecture 45: Friday, November 3. Today, we worked in teams to solve difficult applied optimization problems. This is the final topic that will be covered on Midterm 2!

Lecture 44: Wednesday, November 1. Bennet was a guest lecturer today. We worked through two (involved) problems on applied optimization. We also had a quiz on L'Hôpital's rule. **We will have a GRC session tomorrow!**

Lecture 43: Tuesday, October 31. Our goal today was to introduce applications of finding extrema to real-world problems. Each involved the following steps:

1. Label variables.
2. Identify the function that we want to minimize or maximize (i.e., *optimize*), and write it in terms of one variable.
3. Identify the interval on which we want to optimize the function.
4. Use our mathematical theory developed to optimize the function and solve the problem.

Later this week, we will have a quiz, a GRC session, and practice problems related to the newest topics that will be covered on Midterm 2!

Lecture 42: Monday, October 30. Today, we worked in teams to find limits using creative applications of L'Hôpital's rule.

Lecture 41: Friday, October 27. We started class today by showing that $\lim_{x\to\infty}(x^2-e^x)=-\infty$ by rewriting the function and then applying L'Hôpital's rule to one part.

Next, we turned back to the problem that we started yesterday, of graphing $f(x)=10x^3-x^5$ using calculus methods. Using the first derivative, we found all critical points, determined where the function is increasing/decreasing, all local minima and maxima, and other points with horizontal tangent line. Using the second derivative, we found the concavity of the function and its inflection points. Finally, using the function's zeros, and the values at all transition points, we sketched a possible graph of the function.

After this, we did the same for $g(x)=\sin x+x$ on the interval $[0,2\pi]$. We then turned to the rational function $h(x)=\frac{3x+2}{2x-4}$. Since $x=2$ is not in the domain of h, it is *not* a critical point. However, we include this point as a possible transition point when checking signs of h' and h''. We also needed to find four limits to determine the vertical and horizontal *asymptotes* of h, and its behavior near them.

Lecture 40: Thursday, October 26. We started class by continuing our discussion on the *asymptotic* behavior of functions, and how to make conclusions in this direction using L'Hôptial's rule. Next, we proved this rule in a special case.

After this, we began a detailed discussion on sketching the graph of a function based on data we can obtain by its first and second derivatives. We considered the function $f(x) = 10x^3 - x^5$, and we'll continue this discussion tomorrow.

Lecture 39: Wednesday, October 25. Today, we continued our discussion of **L'Hôpital's rule**, which can be applied to find certain limits. We discussed applying this rule in different ways to find many limits of indeterminate form, including $\frac{0}{0}, 0^0, \infty - \infty$, and $0 \cdot \infty$. Each time, we had to determine first whether the rule applies!

Lecture 38: Tuesday, October 24. We first discussed the **second derivative test for critical points in more detail**. We constructed examples where the test fails in different ways, and used the test to find all local extrema of the function $f(x) = (1 - x - x^2)e^x$.

Next, we noticed that we (still) have no method to find certain limits. Motivated by this, we stated **L'Hôpital's rule** in a first case, and applied it to conclude that

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

Lecture 37: Friday, October 20. Today, we defined what it means for a function f that is differentiable on an open interval to be **concave up** on the interval: f' is in increasing on the interval. Similarly, f is **concave down** if f' is decreasing on an open interval. We saw what this means graphically, and described these concepts in terms of rates of change.

We determined the following **test for concavity** in the case that we can find a second derivative: If f'' exists on an open interval and $f''(x) > 0$ on the interval, then f is concave up there. An analogous statement holds when $f''(x) > 0$ is replaced with $f''(x) < 0$, and "concave up" is replaced with "concave down."

We were motivated by the example of $f(x) = \sin x$, and we found, using its second derivative $f''(x) = -\sin x$, open intervals where the function is concave down and where it is concave up.

Next, we made the following definition: A point $P = (c, f(c))$ on the graph of $f(x)$ is called a **point of inflection**, or **inflection point**, if f changes concavity at $x = c$ (i.e., changes from concave down to concave up, or vice versa).

We noted the points of inflection for $f(x) = \sin x$.

From here, we determined the following **test for inflection points** in the case that we can find a second derivative: If $f''(c) = 0$ or $f''(c)$ does not exist *and* f'' changes sign at $x = c$, then f has a point of inflection at $x = c$.

We used this to determine the points of inflection of $f(x) = 3x^5 - 5x^4 + 1$ and $f(x) = x^{5/3}$.

Finally, we motivated and stated the **second derivative test for critical points**: If c is a critical point of f and $f'(c)$ exists, then

- $f''(c) > 0 \implies f$ has a local minimum at $x = c$,
- $f''(c) < 0 \implies f$ has a local maximum at $x = c$, and

- $f''(c) = 0$ allows us to make no conclusion: it is possible that f either has a local minimum, a local maximum, or neither, at $x = c$.

Remember that on Monday, we will have a GRC session with Bennet.

Lecture 36: Thursday, October 19. Today, we first went over a homework problem. Next, we introduced the **first derivative test** for critical points:

- If $x = c$ is a critical point and $f'(x)$ changes sign from positive to negative at c, then f has a local maximum at c.
- If $x = c$ is a critical point and $f'(x)$ changes sign from negative to positive at c, then f has a local minimum at c.

This is a powerful theorem! For example, this will work even if we are not restricted to the domain being a closed interval.

We applied this in an example, and then in teams, worked through two other problems using the first derivative test. After this, we started some (homework) problems on the guaranteed solution $x = c$ in the Mean value theorem.

Lecture 35: Wednesday, October 18. We started class by stating and proving **Rolle's Theorem**: Suppose that f is continuous on the interval $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists (at least one) c in (a, b) for which $f'(c) = 0$. Its proof used the notion of a critical point, and Fermat's theorem on extrema.

We saw what Rolle's Theorem means in terms of a graph, and found the value c in an example.

We then turned to a generalization of Rolle's Theorem called the **Mean value theorem (MVT)**: Suppose that f is continuous on the interval $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists (at least one) c in (a, b) for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

We interpreted the conclusion geometrically as follows: the tangent line to f at $x = c$ is parallel to the secant line between $(a, f(a))$ and $(b, f(b))$. We saw some graphical examples, and computed the value c for an example given by an equation as well.

Next, we defined what it means for a function f to be **increasing**, **decreasing**, **non-increasing**, and **non-decreasing** on an interval (a, b). A function is **monotonic** on an interval is it is any one of these on this interval.

Using the MVT, proved a theorem that we have intuition for given our knowledge of the relationship between derivatives and tangent lines:

- If $f'(x) > 0$ for all x in (a, b), then f is increasing on (a, b).
- If $f'(x) >\geq$ for all x in (a, b), then f is non-decreasing on (a, b).
- If $f'(x) < 0$ for all x in (a, b), then f is decreasing on (a, b).
- If $f'(x) \leq 0$ for all x in (a, b), then f is non-increasing on (a, b).

Lecture 34: **Friday, October 13**. Today we worked on a good number of problems that involved finding extrema on closed intervals.

Lecture 33: **Thursday, October 12**. Today, we defined **extreme values**, or **extrema** on an interval; these are **minimum** and **maximum** output values of a function among all input values on an interval.

Suppose that a function f is defined on an interval I. We noticed that if either (1) f is not continuous on I, or (2) I is not a closed interval (e.g., it is infinite), then it is possible that f has no minimum or maximum value on the interval.

Then we proved the following **theorem**: If f is continuous on a closed interval $I = [a, b]$, then f has a minimum and a maximum on I.

We moved on to the question of *how* to find the minimum and maximum values on an closed interval, in the case that this theorem ensures that they must exist. To do so, we defined **critical points** if f on a closed interval I as points $x = c$ for which one of the following hold:

1. $f'(c) = 0$
2. f is not differentiable at c
3. c is an endpoint of I (which can be thought of a special case of (2)).

Then we stated and proved the amazing **Fermat's theorem**: If $f(c)$ is a local minimum or maximum, then f is a critical point of f.

We then applied this theorem in several examples, finding the minimum and maximum of a continuous function on a closed interval.

Lecture 32: **Wednesday, October 11**. We discussed the *linearity* of the definite integral, and then turned to linearization.

We motivated the fact that near an x-value a, if $f(x)$ is differentiable at a, then the output of the tangent line to $f(x)$ at $x = a$ is an approximation for the output value of f. We compared these estimates, and used the ideas to estimate $\ln(1.1)$ and $\sqrt{4.1}$. We also applied this idea in a "real-world" example.

Lecture 31: **Tuesday, October 10**. Today, we first worked on a homework problem evaluating a sum in sigma notation.

Next, we defined the **definite integral** of a function f on an interval $[a, b]$ (assuming that it is continuous on this interval). as the limit as N approaches infinite of an arbitrary sequence of Riemann sums with N rectangles. This definite integral is denoted

$$\int_a^b f(x)\,dx,$$

and is a number; the **area** under the graph of f from $x = a$ to $x = b$.

We found that $\int_a^b C\,dx = C(b-a)$, which means that $\int_a^b (-4)\,dx = -4(b-a)$, a negative number. This motivated the fact that for consistency, we consider the integral a *signed* area, where the area below the x-axis counts as negative area. Moreover, if $a < b$, we use the convention that $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

We went through several examples of finding a definite integral exactly using geometric information, and estimating or finding the sign of such an integral.

We deduced some properties of the definite integral. For example, if $f(x) \leq g(x)$ for all x in $[a,b]$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

If $m \leq f(x) \leq M$ for all x in $[a,b]$, we have that

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Lecture 30: Monday, October 9. Today we worked in groups calculating Riemann sums, and then finding their limits to evaluate the area under a graph.

Lecture 29: Friday, October 6. We started class by recalling the notation for Riemann sums, and the theorem stating that the limit of each of these sums as the number of rectangles approaches infinity is the area under the graph. We then recalled sigma notation, and derived the formula

$$\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

in two different ways. Note that there are analogous formulas for $\sum\limits_{i=1}^{n} i^2$ and $\sum\limits_{i=1}^{n} i^3$; they appear, for example, in your textbook (and we can prove them using induction!).

Next, we calculated the area under two graphs on the interval $[0,6]$ by first finding the general form of an N^{th} Riemann sum, and taking the limit $N \to \infty$. One graph was a line, so we could verify the answer by calculating the area geometrically, and the second was a parabola.

Lecture 28: Thursday, October 5. Today we introduced the concept of the area under the graph of a function. Given a function $f(x)$ that is continuous and has non-negative values on the interval $[a,b]$, we can estimate the area under the graph of f between $x = a$ and $x = b$ using the areas of rectangles, which are called **Riemann sums**. If the rectangles all abut the graph on the right, we call the sum of the areas the **right-hand sum**, and if they touch on the left, we call this the **left-hand sum**. If there are N rectangles of equal width, we call the width of each rectangle Δx, and compute that this equals $\frac{b-a}{N}$. We call the right-hand x-coordinate of the k^{th} rectangle x_k, so that $x_N = b$, and calling $x_0 = a$ is consistent. The right-hand sum with N rectangles equals

$$R_N = \sum_{k=1}^{N} f(x_k)\Delta x = \Delta x \cdot (f(x_1) + f(x_2) + \cdots + f(x_N)),$$

and the left-hand sum with N rectangles equals

$$L_N = \sum_{k=0}^{N-1} f(x_k)\Delta x = \Delta x \cdot (f(x_0) + f(x_1) + \cdots + f(x_{N-1})).$$

We can derive similar formulas for sums where the rectangles touch the graph at the midpoint of an interval, or where we replace rectangles with trapezoids touching the graph in two places.

We did some example, and noticed that by taking more an more rectangles, we appear to be getting better and better estimates for the area under the graph. We stated an **important theorem**, which says that the limit of each of the approximations, as the number of rectangles approaches infinity (so that the width Δx approaches zero), equals the same finite number in our setup. In particular,

$$\lim limits_{N\to\infty} R_N = \lim_{N\to\infty} L_N = L,$$

where L is the area under the graph of $f(x)$ between $x = a$ and $x = b$.

Lecture 27: Wednesday, October 4. We had a GRC session today with Bennet, on functions between sets, including the concepts of *one-to-one*, *onto*, and *bijections*.

Lecture 26: Monday, October 2. Today, we worked on some challenging problems related to Midterm 1 material. This included related rates, graphs of derivatives and higher derivatives, deciding continuity and differentiability of functions, and identifying when to use logarithmic differentiation.

Lecture 25: Friday, September 29. Today, we solved two related rates problems together, which involved different setups, and different ways to relate the functions that we need to relate. In each, the general process is the following:

1. Draw a diagram if necessary.
2. Define variables and functions involved in the problem, and label your diagram appropriately.
3. Identify the values and rates that are known, and that which are desired.
4. Find a relationship between the known and desired rates.
5. Write an equation only involving the functions from (3).
6. Take derivatives of both sides of the equation from (5), and solve for the desired value.

Note that steps (4) and (5) may involve, for example, using similar triangles or the Pythagorean theorem.

Next, we worked on two new problems in teams.

Lecture 24: Thursday, September 28. Today, we started class with a (slightly complicated) homework problem requiring us to find specific points on an implicit function, and the derivative of each of these points.

Next, we turned to applying derivatives to the "real world" by interpreting them as *rates of change*. Our first example is a function that outputs the position of an object at a certain time; its derivative is the instantaneous velocity! We then turned to finding how the area of a circle grows as its radius grows.

Next, we set up a more involved problem. Given a ladder leaning on a wall, if we are given its length, the inital distance of the foot of the ladder from the wall, and the rate at which it moves from the wall, we are able to find the velocity at which the top of the ladder slides down the wall. Indeed, we created functions $x = x(t)$ giving the distance of the foot from the wall at time t, and $y = y(t)$ the distance of the top of the ladder to the floor at time t. We interpreted the desired rate and the given rate as derivatives. To find one in terms of the other, we related the functions: $x^2 + y^2$ is the square of the length of the ladder. Next, we differentiated this equation with respect to t, and were able to solve for $\frac{dy}{dt}$ in terms of $\frac{dx}{dt}$.

Lecture 23: Wednesday, September 27. Today, we first found the derivative of $\arctan x$ using the fact that it is an inverse of $\tan x$, when tangent is restricted to $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Next, we turned to finding limits involving trigonometric functions. We showed that the function

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable at $x = 0$, and $f'(0) = 0$!

To do so, we used the **Squeeze theorem** for limits: If $g(x) \leq f(x) \leq h(x)$ and $\lim_{x \to c} g(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} f(x) = L$ as well. Here, we used the fact that $-1 \leq \sin(y) \leq 1$ for all values y; e.g., any $y = \frac{1}{h}$. Therefore, $-1 \leq \sin\left(\frac{1}{h}\right) \leq 1$, and $\frac{-1}{h} \leq \frac{\sin\left(\frac{1}{h}\right)}{h} \leq \frac{1}{h}$ for all $h \neq 0$ (check both $h > 0$ and $h < 0$!), and this middle term appears in the limit used to define the derivative at $x = 0$.

We then motivated why

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \quad \text{and} \quad \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0.$$

We used the first to find $\lim\limits_{x\to 0} \frac{\sin(4x)}{x}$ and $\lim\limits_{x \to 0} \frac{\sin(4x)}{\sin(5x)}$, by substitution (e.g., $\theta = 4x$) and/or rewriting the function to look more like the limits above.

Tomorrow we will start our last topic covered on Midterm 1, related rates! We will also have a quiz on implicit differentiation and/or finding derivatives of inverse functions.

Lecture 22: Tuesday, September 26. In teams, today we worked on problems involving implicit differentiation and finding derivatives of inverse functions.

Lecture 21: Monday, September 25. We started with a challenging problem involving implicit differentiation, finding the points on the curve

$$(x^2 + y^2)^2 = x^2 - y^2,$$

where the tangent line is horizontal.

After this, we discussed inverse trigonometric function via the example $y = \arcsin x$ (called $\sin^{-1} x$ in the text), and then used implicit differentiation to find its derivative, differentiating $x = \sin y$ with respect to x.

Next, we used an analogous method to find the derivative of $y = \ln x$ by using the identity

$$e^{\ln x} = x,$$

and differentiating both sides. (This identity holds because $y = e^x$ and $y = \ln x$ are inverse functions, right?) We applied this rule in some examples.

Finally, we showed that $\frac{d}{dx}(a^x)$ is, in fact, equal to $a^x \cdot \ln a$ by writing $y = a^x$, taking the natural logarithm of both sides, and then implicitly differentiating.

Lecture 21: Friday, September 22. Today, we had a GRC session with Bennet that dealt with derivatives, combined with the application of induction.

Lecture 21: Thursday, September 21. Today, we introduced **implicit functions**, which are the solutions to equations involving x and y, but for which we may, or may not, be able to solve for y as a function of x. The natural first example is the equation for the unit circle:

$$x^2 + y^2 = 1.$$

To turn this into the form $y = \cdots$, we actually need *two* actual functions, $y = \sqrt{1 - x^2}$ and $y = \sqrt{1 - x^2}$.

How do we find the general derivative? This is the method of **implicit differentiation**, where we treat y as a function of the variable x *locally.* This means that, in particular, $\frac{d}{dx}y = \frac{dy}{dx}$, but $\frac{d}{dx}x = 1$. Therefore, when implicitly differentiating both sides of an equation, we must apply the chain rule whenever encountering "y."

For the circle above, in taking the derivative of both sides with respect to x, we find:

$$2x + 2y\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$, we see that

$$\frac{dy}{dx} = -\frac{x}{y}.$$

So, for example, we verify that the slope of the tangent line at the point $(0, -1)$ on the unit circle equals $-\frac{0}{-1} = 0$, and at $\left(\sqrt{2}/2, \sqrt{2}/2\right)$ is 1. At $(1, 0)$, the fraction $1\frac{1}{0}$ is undefined, and so is its derivative: we see that the tangent line to the curve at the point is vertical.

We did two more examples, which were increasing in difficulty. For fun, graph our equation

$$e^{x-y} = \sin(xy)$$

using a computer program or website, and try to find a point lying on it! We found an equation for the derivative of *every point* on this curve!

Tomorrow we will have a GRC session with Bennet! Make sure to make it – we will have assignments on this material soon.

Lecture 20: Wednesday, September 20. We worked through several non-traditional examples of applying the derivative rules that we've encountered. Time permitting, we will briefly introduce implicit differentiation.

There will be a quiz tomorrow. Make sure to complete the homework problems!

Lecture 19: Tuesday, September 19. We started class by motivating the derivative of $y = sinx$ using graphs. In fact,

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = \sin x.$$

We will soon be able to prove these!

Using these formulas and the quotient rule, we found the derivatives of $\tan x$ and $\sec x$; you'll do $\cot x$ and $\csc x$ as exercises (one is a homework problem).

This motivated the fact that we might want to find the derivative of $sin(2x)$. We recalled that yesterday we found the derivatives of e^{2x}, e^{3x}, and e^{-x}, and all of these functions of the form e^{cx} ended up having derivatives ce^{cx}. However, our method of interatively applying the product rule won't work for something like $e^{\pi x}$.

We stated the **chain rule** for derivatives: If g is differentiable at x, and f is differentiable at $g(x)$, then

$$\frac{d}{dx}f(g(x)) = (f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

Using Liebniz notation, if $y = g(x)$ and $z = f(y)$, then

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}.$$

We applied the chain rule in several examples. In each, we needed to identify the function as the composition on two functions, and then find the derivative of each, before plugging them into the chain rule formula.

Lecture 18: Monday, September 18. In groups, today we worked through several problems involving the product and quotient rules. Many of these had a different flavor than just "applying" the rules to formulaic equations.

Lecture 17: Friday, September 15. We began class by finding the derivative of $f(x) = \frac{1}{\sqrt{x}}$ using the definition of the derivative, a homework problem.

Next, we proved the theorem from last time: If a function is differentiable at a point, it must be continuous at that point.

After this, we introduced the **product rule** for derivatives: if $f(x)$ and $g(x)$ are differentiable at x, then

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

We did several examples applying this rule, and then proved that it holds.

Finally, we introduced the **quotient rule** for derivatives: if $f(x)$ and $g(x)$ are differentiable at x, and $g(x) \neq 0$, then

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

You'll practice applying this rule in homework, and we will work on some new ways of applying the product and quotient rules in class on Monday.

Lecture 16: Thursday, September 14. We started class today by completing a homework problem, finding the derivative of $f(x) = x - \sqrt{x}$ using the limit definition. Next, we recalled the derivatives we showed yesterday, and then proved the second part of **linearity of the derivative**, assuming that f and g are differentiable at x:

1. If c is a constant, then $(cf)'(x) = cf'(x)$.
2. $(f+g)'(x) = f'(x) + g'(x)$ and $(f-g)'(x) = f'(x) - g'(x)$.

We next proved that if $n > 0$ is an integer,

$$\frac{d}{dx}(x^n) = nx^{n-1}. \tag{1}$$

This used the fact that

$$(x^n - a^n) = (x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1}).$$

In fact, the derivative formula (1) is true for *any* nonzero real number! This is typically called the **power rule** for derivatives. We did some examples in applying this, and linearity.

After this, we showed that for any $a > 0$, the derivative of a^x is a constant multiple of a. In fact, this multiple is $\ln a$, so that

$$\frac{d}{dx}(a^x) = (\ln a)a^x, \text{ and}$$
$$\frac{d}{dx}(e^x) = (\ln e)e^x = e^x.$$

Finally, we stated an important **Theorem**: If a function is differentiable at a point, then it must be continuous at that point!

We noted that the converse does *not* hold; i.e., there are functions that are continuous, but not differentiable, at points. Functions can fail differentiability in several ways: the limit in the definition of the deriviative does not exist for several reasons, or it is infinite. We provided several examples illustrating these notions.

Lecture 15: Wednesday, September 13. We started class today by proving another statement by mathematical induction.

Next, we discussed the **derivative function**: Given a function f, we create a new function f'. Its domain consists of all real numbers $x = c$ for which the derivative $f'(c)$ exists and is finite. For these values, this derivative function is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

We found that if $f(x) = |x|$, then the derivative function has domain all real numbers except zero, and

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Using the definition of the derivative function, we found that the derivative of any constant is zero, and the derivative of every line is its slope. For each, we were motivated by the graph of such a function, and verified that our conjectured derivative was correct using the definition.

Next, we found that the derivative of $f(x) = x^2$ is $f'(x) = 2x$ from definition, and then started to find the derivative of $g(x) = \frac{1}{x^2}$. From its graph, we found that the derivative function should have domain all real numbers except zero, should be positive for $x < 0$, and should be negative for $x > 0$. If you didn't finish finding this, do it before class tomorrow!

Lecture 14: Tuesday, September 12. We had our second GRC session with Bennet, on the **principle of mathematical induction.** Remember to bring your GRC worksheet to class tomorrow!

Lecture 13: Monday, September 11. We started class today by proving that there exists a line that bisects both a slice of ham, and the slice of bread it lies on, simultaneously! This used the Intermediate Value Theorem, and assumed the continuity of certain function. This reasoning is the start to the proof of the Ham Sandwich Theorem!

Next, we reviewed our discussion of derivatives at a point, including the graphical interpretation involving slopes of secant and tangent lines. Using the definition of a derivative, we calculated derivative of several functions, and determined the sign of others.

In particular, we saw that the derivative of a function f at a point $x = a$ need not exist, in the case that the limit in the definition,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

does not exist. For example, the derivative of $\alpha(x) = |x|$ does not exist at $x = 0$ since $\lim_{h \to 0^+} \frac{|h|}{h} = 1$, while $\lim_{h \to 0^+} \frac{|h|}{h} = -1$.

Moreover, we saw that it is not always easy to find the limit appearing in the definition of a derivative. For example, if $E(x) = e^x$, then

$$E'(1) = \lim_{h \to 0} \frac{e^{1+h} - e^h}{h} = \lim_{h \to 0} \frac{e^h(e-1)}{h}$$

if this limit exists, but what is this number! (Many of you already know some derivative rules, and claimed it equals e. But why?!)

We will have a GRC session with Bennet tomorrow!

Lecture 12: Friday, September 8. We started class by applying the IVT to show that a certain number has an 11-th root. You'll do this more generally for homework!

Next, we went back to our discussion of the Ham Sandwich Theorem.

From here, we turned to the definition of a derivative. We motivated the definition of a tangent line to a function $f(x)$ at a point a via the limit of secant lines, drawing several graphs. The **derivative of $f(x)$ at a** is

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h},$$

and is the slope of the tangent line to $f(x)$ at a.

Lecture 11: Thursday, September 7. After covering a homework problem, we reviewed the Intermediate Value Theorem, and then applied it to show that there is a solution to the equation

$$e^x = x^2,$$

and used the bisection method to estimate a solution in the interval $(-1, 0)$.

Next, we worked in teams to solve some challenging problems that involved the IVT. Finally, we started discussing the Ham Sandwich Theorem.

Lecture 10: Wednesday, September 6. We started class by going over several homework problems, and some additional challenge problems on limits.

Next, we motivated and stated the **Intermediate Value Theorem** (IVT): If f is a continuous funcion on a closed interval $[a, b]$ and M is any value between $f(a)$ and $f(b)$, then there is at least one value c in the interval (a, b) for which

$$f(c) = M.$$

We gave several examples where the theorem holds, and where it does not. Then we started to show the existence of solutions to equations by applying the IVT.

Lecture 9: Tuesday, September 5. Today, we defined limits as the input value x approaches ∞ and $-\infty$ rigorously. We then showed that if $n > 0$ is a real number, then $\lim\limits_{x\to\infty} x^n = \infty$ and $\lim\limits_{x\to\infty} \frac{1}{x^n} = 0$. We used this, motivated by intuition, to find several limits, many of which required factoring out "leading terms" from a numerator and denominator.

Lecture 8: Friday, September 1. We worked through a homework problem, and then a bunch of problems that involved determining whether a limit exists, and finding the limit if it does. Many of these were of indeterminate form, and some required taking limits from both sides.

Lecture 7: Thursday, August 31. We started class by studying in more detail a problem from the GRC session yesterday, which involved investigating the limits

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right).$$

Next, we investigated a few limits in order to illustrate the following **theorem**: If f is a function that agrees with a continuous function g at all points of an open interval containing $x = c$, except possibly at c itself, then

$$\lim_{x\to c} f(x) = g(x).$$

Next, we discussed limits of functions of **indeterminate form**: Where the function is written as a quotient, product, or difference so that the limit laws cannot immediately evaluate it. We abbreviate these as:

$$\frac{\infty}{\infty}, \frac{0}{0}, \infty \cdot 0, \infty - \infty.$$

We developed several methods that help find these limits (or determine that they do not exist). In many cases, this includes applying the above theorem after rewriting the equation of the function appearing in the limit. The rewriting can take several forms: factoring, rationalizing, and combining fractions via a common denominator.

We ended with a few hard problems that we'll come back to tomorrow.

Lecture 6: Tuesday, August 29. Today, we classified discontinuities, defining a **removable**, **jump**, and **infinite** discontinuity. A discontinuity can have more than one type!

We recalled the Basic Laws of Continuity, and added some slightly less basic ones. For example, if $g(x)$ is continuous at $x = c$ and $f(x)$ is continuous at $x = g(c)$, then the composite function

$$(f \circ g)(x) = f(g(x))$$

is continuous at $x = c$.

Additionally , all polynomials are continuous at all real numbers, and rational functions (quotients of polynomials) are continuous everywhere where the denominator is nonzero. The functions $y = \sin x$, $y = \cos x$, and $y = a^x$ are continuous at all real numbers if $a > 0$. If n is a positive integer, the function $y = x^{1/n}$ is continuous on its entire domain.

We did a few example, and then applied our theorems to find limits, using the fact that a function is continuous precisely if $\lim_{x\to c} f(x) = f(c)$.

Lecture 5: Monday, August 28. We started class by working through a homework problem.

Our main goal today was to define what it means for a function to be continuous at a point, and understand it. Suppose that a function $f(x)$ is defined on an open interval containing $x = c$. We say that $f(x)$ **is continuous at** c if

$$\lim_{x \to c} f(x) = f(c).$$

We noticed that to have *any hope* that $f(x)$ is continuous at c, we need $f(c)$ to be defined (i.e., c is in the domain of f), $\lim_{x \to c} f(x)$ must exist and be finite, and then these two values, $f(c)$ and the limit, must coincide. We gave several types of examples where each of these fail, using graphs.

Next, we used the definition of continuity to decide where several functions that are defined by equations have discontinuities, including a piecewise function. In particular, we showed that every constant function, and every function of the form x^n, where n is a positive integer, are continuous at all real numbers.

We finished class by stating several **Basic Laws of Continuity**, and saw how they can be applied to show that more complicated functions are continuous at a point. These laws can be shown using the Limit Laws.

Lecture 4: Friday, August 25. Today we started with a problem from homework. Next, we discussed the Limit Laws, which allow us (sometimes!) to skip over the δ-ε definition of a limit. We gave several examples, which culminated in defining what it means for a limit to be infinite. Our definition is as follows: If $f(x)$ is defined on an open interval containing $x = c$, except possibly at $x = c$ itself, then the $f(x)$ **approaches infinity as** x **approaches** c,

$$\lim_{x \to c} f(x) = \infty,$$

if for every $N > 0$, there is a distance $\delta > 0$ for which $f(x) > N$ whenever $|x - c| < \delta$, i.e., x is within δ of c. There is an analogous definition for $\lim_{x \to c} f(x) = -\infty$ (i.e., $f(x)$ can be made arbitrarily negative by making x sufficiently close to c.

We finished class by creating functions that satisfied several limit conditions, and compared our answers.

Lecture 3: Thursday, August 24. Our class began by formalizing the definition of a limit. If f is a function defined on an open interval containing $x = c$, except possibly at c itself, then we say that the **limit of** $f(x)$ **as** x **approaches** c **is** L, and write

$$\lim_{x \to c} f(x),$$

if the following holds: Given any distance $\varepsilon > 0$, there is some distance $\delta > 0$ such that whenever x is within δ of c, i.e., $|x - c| < \delta$, then $f(x)$ is within ε of L, i.e., $|f(x) - L| < \varepsilon$.

We sketched a graph to illustrate this concept, and then investigated the values of the distance δ based on the given distance ε in showing that $\lim_{x \to 4} \frac{1}{2}x = 2$. For $\varepsilon = 2$, we calculated $\delta = 1$, and after trying some more values, we conjectured that given any $\varepsilon > 0$, the distance $\delta = 2\varepsilon$ should work in the definition. Then we verified equationally: If $|x - 4| < \delta = 2\varepsilon$, then $\frac{1}{2}|x - 4| < \varepsilon$, so that $|f(x) - 2| = \left|\frac{1}{2}x - 2\right| < \varepsilon$!

From here, we worked in teams to find appropriate δ values given a distance ε for the following limits: $\lim_{x\to 3} x = 3$ ($\delta = \varepsilon$ works), $\lim_{x\to -1}(2x+1) = -1$ ($\delta = \frac{1}{2}\varepsilon$ works), and $\lim_{x\to 3} 5 = 5$ (any $\delta > 0$ works!).

Next, we pointed out that we probably cannot do this process all the time, so we'll need to establish some shortcuts to finding limits. We'll state these "Limit Laws" tomorrow. Make sure to familiarize yourself with them before class!

Lecture 2: Wednesday, August 23. We started class today by graphically investigating limits. We noticed that a limit $\lim_{x\to c} f(x)$ can exist even if c is not in the domain of the function $f(x)$!

If $f(x)$ is a function defined on an open interval containing $x = c$, except possibly not defined at $x = c$, then we say that the **limit of $f(x)$ as x approaches c is L**, and write

$$\lim_{x\to c} f(x),$$

if we can make $f(x)$ arbitrarily close to L by choosing x sufficiently close to c. Soon, we will turn this statement into a more formal mathematical one.

We investigated the limits $\lim_{x\to 1} e^x$, $\lim_{x\to 0} \frac{\sin x}{x}$, and then in groups, $\lim_{x\to 0} \cos\left(\frac{1}{x}\right)$. For the last one, we noticed that making a table can be deceiving!

Lecture 1: Tuesday, August 22. We started the course by investigating the course website, discussing the expectations and syllabus in detail, and giving some overview.

Next, we pointed out the fundamental difference between calculus and its precursors: **limits**. These are motivated by, and intertwined with, the concept of **rates of change**. We defined the **average rate of change** of a function over an interval, and the **instantaneous rate of change** of a function at a point through the example of velocity; i.e., by studying the function whose input is time, and whose output is position. Both are the slopes found via the graph of the function: the first is a slope of a *secant line* through two points, and the second is the slope of the *tangent line* to a point on the graph. The second requires a limit, while the first does not!