LOCAL COHOMOLOGY OF SUBSPACE ARRANGEMENTS AND SIMPLICIAL HOMOLOGY

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Abstract. This paper provides a new understanding of the local cohomology modules of a polynomial ring R over a field with support in an ideal I that defines a central linear subspace arrangement. We define a family of simplicial complexes associated to a ring, whose faces are determined by the way that the irreducible components of the ring's spectrum fit together. Carrying out an analysis of the Mayer-Vietoris spectral sequence in local cohomology, we prove that the relative simplicial homology of pairs of these complexes govern the vanishing of local cohomology of R with support in I. Moreover, the (relative) homology of the complexes provides further concrete information about the structure, and Bass numbers, of these modules. In particular, we relate the Lyubeznik numbers of the localization S of R/I at its homogeneous maximal ideal to these values, and as a consequence, establish a formula for the Lyubeznik number $\lambda_{0,2}(S)$. Moreover, we use the theory developed to relate the dimension of the support of the local cohomology to connectedness dimension.

1. Introduction

The overarching goal of this paper is to advance the understanding of local cohomology modules with support in an ideal defining a central linear subspace arrangement, i.e., an ideal of a polynomial ring whose minimal primes are generated by linear forms. In particular, we design a family of simplicial complexes whose simplicial homology relays intrinsic information about the local cohomology. Our technique works over fields of arbitrary characteristic, and takes advantage of this framework to carry out a systematic analysis of the Mayer-Vietoris spectral sequence in local cohomology.

Local cohomology with support in ideals defining a linear subspace arrangement, especially in the special case of monomial ideals, has a rich history of study; e.g., see [Ter98, Mus00, EMS00, Yan01, ÀGZ03, ÀV14, DJ15, ÀY18]. Our work provides a new perspective that illustrates a deep connection between these local cohomology modules and the simplicial homology of a family of simplicial complexes. These so called *codimension complexes*, introduced in Definition 2.1. reflect the way that the irreducible components of the associated variety connect to one another. The method we develop yields a concrete characterization of when the local cohomology vanishes, and provides new ways to understand its structure.

Definition A. Suppose that $P_1, \dots, P_\ell$ are the minimal primes of a d-dimensional Noetherian ring S. Given an integer $t \geq 0$, the *t-th codimension complex*, denoted $\Lambda_t(S)$, is the simplicial complex containing $\varnothing$, whose other faces are defined as follows:

$$\sigma \subseteq \{1, 2, \dots, \ell\} \text{ is a face of } \Lambda_t(S) \text{ if and only if } \dim(S/J) \geq d - t,$$

where J is the sum of all P_i for which $i \in \sigma$. Moreover, let $\Lambda_{-1}(S) = \{\varnothing\}$.

Equivalently, if $V_1, \dots, V_\ell$ denote the irreducible components of the prime spectrum of S, then the faces of $\Lambda_t(S)$ are all $\sigma \subseteq [\ell]$ for which the dimension of $\cap_{i\in\sigma} V_i$ is at least $d-t$.

The family of codimension complexes encompasses different tools that have been used to study local cohomology. For instance, suppose that $(S, \mathfrak{m})$ is a complete, equidimensional, local ring containing a field. Then $\Lambda_{d-1}(S)$ is the complex used by Lyubeznik to investigate cohomological dimension [Lyu07], which, in this setting, has vertex set $\{1, 2, \dots, \ell\}$, and $\{i_1, \dots, i_j\}$ is defined to be a face when $P_{i_1} + \cdots + P_{i_j}$ is not $\mathfrak{m}$-primary. Subsequently, he, Katzman, and Zhang also related its homology to the depth of S [KLZ16].

In this same setting, the 1-skeleton of $\Lambda_1(S)$ is the *Hochster-Huneke graph*, also referred to as *(Hartshorne's) dual graph*, of S, which has been used to study connectedness properties of spectra and Lyubeznik numbers [Har62, HH94, Lyu06, Zha07, HNPW19]. The second author, with Núnez-Betancourt and Spiroff, studied the 1-skeletons of other $\Lambda_t(S)$, and used that of $\Lambda_{d-2}(S)$ to obtain a "Third Vanishing Theorem" for local cohomology [NSW19].

Given a polynomial ring R over a field $\Bbbk$, suppose that an ideal I of R defines a linear subspace arrangement. In Theorems 5.1 and 5.6, we characterize the vanishing of each local cohomology module $H^i_I(R)$, and the dimension of its support $\operatorname{Supp} H^i_I(R)$, respectively, in terms of the vanishing of certain simplicial homology modules of the $\Lambda_t(R/I)$ relative to their subcomplexes $\Lambda_{t-1}(R/I)$, with coefficients in $\Bbbk$.

Theorem B. *Let R be an n-dimensional polynomial ring over a field $\Bbbk$, and I an ideal of R of height h that defines a central linear subspace arrangement.*

(1) *Given any integer i, $H^i_I(R) = 0$ if and only if for every $0 \le j \le n-i$,*
$$H_j(\Lambda_{i+j-h}(R/I), \Lambda_{i+j-h-1}(R/I); \Bbbk) = 0.$$
(2) *Moreover, if $H^i_I(R)$ is not zero, then the dimension of its support equals*
$$n - i - \min\{0 \le j \le n-i \mid H_r(\Lambda_{i+j-h}(S), \Lambda_{i+j-h-1}(S); \Bbbk) \ne 0\}.$$

Theorem B (1) implies that the cohomological dimension $\operatorname{cd}(R, I)$ of I, i.e., the largest $i \in \mathbb{Z}$ for which $H^i_I(R) \ne 0$, is determined solely by the vanishing of the reduced simplicial homology of the $\Lambda_t(R/I)$ with coefficients in $\Bbbk$. This formula, appearing as Corollary 5.2 in the paper, and as the first statement in the following corollary, when combined with the characterization of the depth of a Stanley-Reisner ring in terms of simplicial homology [DDD+19], provides a new proof of the classical result relating the depth of R/I and $\operatorname{cd}(R, I)$, originally proved by Lyubeznik using resolutions [Lyu84]; see Theorem 8.1.

Corollary C. *Let $R = \Bbbk[x_1, \dots, x_n]$, where $\Bbbk$ is a field, and fix an ideal I of R that defines a central linear subspace arrangement. Then the following hold.*

(1) *If $\dim(R/I) = d$, then $\operatorname{cd}(R, I) = n - \min\{i + j \mid \widetilde{H}_j(\Lambda_{d-i}(R/I); \Bbbk) \ne 0\}$.*
(2) *If I is a squarefree monomial ideal, then $\operatorname{cd}(R, I) = n - \operatorname{depth}(R/I)$.*

In a different direction, we take advantage of Theorem B (2) to establish, in Theorem 5.11, a relation between the local cohomology modules of R with support in I and the connectedness dimension of R/I, the minimum dimension among all closed sets of its spectrum whose complement is disconnected, which does not reference the codimension complexes nor simplicial homology.

Theorem D. *Let I be an ideal of an n-dimensional polynomial ring R over a field, that defines a central linear subspace arrangement. Assume that $\dim(R/I) \geq 2$, and fix a positive integer j such that $j \leq \dim(R/P)$ for any minimal prime P of I. If*

$$\dim \operatorname{Supp} H_I^{n-i}(R) \leq i-2$$

for each $1 \leq i \leq j$, then the connectedness dimension of S is at least j.

Suppose that I is an ideal of a polynomial ring $R = \mathbb{k}[x_1, \ldots, x_n]$, where $\mathbb{k}$ has characteristic zero. Recall that the local cohomology modules of R with support in I each have finite length as a module over the Weyl algebra $\mathcal{D}(R, \mathbb{k})$, the noncommutative ring $\mathbb{k}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$ modulo the two-sided ideal generated by, for $1 \leq i, j \leq n$, the elements $x_i x_j - x_j x_i$, $\partial_i \partial_j - \partial_j \partial_i$, and $\partial_i x_i - x_i \partial_i - 1$, and for $i \neq j$, $\partial_i x_j - x_j \partial_i$ [Lyu93]. In Theorem 5.14, we determine this length for ideals defining central linear subspace arrangements; we use $h_\bullet(-;\mathbb{k})$ to denote $\dim_{\mathbb{k}}(-;\mathbb{k})$.

Theorem E. *Suppose that I is an ideal of a polynomial ring R over a field $\mathbb{k}$ of characteristic zero that defines a central linear subspace arrangement. For each integer i, the length of $H_I^i(R)$ as a $\mathcal{D}(R, \mathbb{k})$-module is*

$$\sum_{j=0}^{n-i} h_j(\Lambda_{i+j-h}(S), \Lambda_{i+j-h-1}(S); \mathbb{k}).$$

Our general technique is built around an in-depth study of the Mayer-Vietoris spectral sequence in local cohomology [ÀGZ03, Lyu07], and works independent of characteristic. If $I_1, \ldots, I_\ell$ are ideals of a Noetherian ring R and I is their intersection, this spectral sequence has the form

$$E_1^{-p,q} = \bigoplus_{1 \leq i_0 < \cdots < i_p \leq \ell} H^q_{I_{i_0} + \cdots + I_{i_p}}(R) \underset{p}{\Rightarrow} H_I^{q-p}(R).$$

When R is a polynomial ring over a field $\mathbb{k}$ and I defines a central linear subspace arrangement, after replacing I with its radical, one can take the I_i to be the minimal primes of I. Since each minimal prime is generated by linear forms, every sum of minimal primes is again generated by linear forms. Hence, the terms on the E_1 page are straightforward to understand, and in fact, the differentials between them are closely related to the maps in the complex used to compute simplicial homology. Investigating this relationship in detail is key to our technique. Moreover, we call upon the fact that the spectral sequence converges at the E_2 page in our setting [ÀGZ03].

Our method also provides a better understanding of the *Lyubeznik numbers* corresponding to central linear subspace arrangements. Recall that if $T \cong A/J$, where J is an ideal of an n-dimensional regular local ring A containing a field, then the Lyubeznik number $\lambda_{ij}(T)$ is the i-th Bass number of $H_J^{n-j}(A)$, i.e., the number of copies of the injective hull of $\mathbb{k}$ in the i-th term of a minimal injective resolution of this local cohomology module. Theorem 6.10 establishes a relationship between Lyubeznik numbers and the $\mathbb{k}$-vector space dimension of the simplicial homology of $\Lambda_t(T)$ relative to $\Lambda_{t-1}(T)$ with coefficients in $\mathbb{k}$.

Theorem F. *Suppose that I is an ideal of a polynomial ring R over a field $\mathbb{k}$ that defines a central linear subspace arrangement. Suppose further that R/I has dimension d, let T*

denote its localization at its homogeneous maximal ideal. Then for any integer $0 \leq j \leq d$*,*

$$\sum_{i=0}^{j}(-1)^i\lambda_{ij}(T) = \sum_{i=0}^{j}(-1)^i h_{j-i}(\Lambda_{d-i}(T), \Lambda_{d-i-1}(T); \Bbbk).$$

As a consequence of this result, we obtain a concrete description of the Lyubeznik number $\lambda_{02}(T)$ in our setting, which depends only on $\Lambda_{d-1}(T)$ and $\Lambda_{d-2}(T)$, and more generally, prove that when T is equidimensional of dimension 3, all Lyubeznik numbers of T depend only on the codimension complexes of T; see Theorem 6.14 and Corollary 6.16.

Corollary G. *Suppose that* I *is an ideal of a polynomial ring* R *over a separably closed field* $\Bbbk$ *that defines a central linear subspace arrangement. Suppose further that* R/I *is equidimensional of dimension* $d \geq 3$*, and let* T *denote its localization at its homogeneous maximal ideal. Then*

$$\lambda_{12}(T) = h_0(\Lambda_{d-2}(T); \Bbbk) - h_0(\Lambda_{d-1}(T); \Bbbk), \text{ and}$$
$$\lambda_{02}(T) = h_0(\Lambda_{d-2}(T); \Bbbk) - h_0(\Lambda_{d-1}(T); \Bbbk) + h_1(\Lambda_{d-1}(T); \Bbbk) - h_1(\Lambda_{d-1}(T), \Lambda_{d-2}(T); \Bbbk).$$

Moreover, if $d = 3$*, then all Lyubeznik numbers of* T *depend only on the codimension complexes* $\Lambda_1(T)$ *and* $\Lambda_2(T)$*.*

The paper is organized as follows: In Section 2, we introduce the family of codimension complexes associated to a ring, and establish some basic properties about their simplicial homology. In Section 3, we decompose the relative simplicial homology with respect to consecutive pairs of codimension complexes in a manner that proves convenient in carrying out a detailed study of the Mayer-Vietoris spectral sequence in local cohomology corresponding to a central linear subspace arrangement, which is the goal of Section 4. Section 5 takes advantage of this to better understand the local cohomology modules of focus, and in Section 6, the Lyubeznik numbers. Finally, Section 7 details some examples illustrating the results from Sections 5 and 6.

2. Codimension Simplicial Complexes

The following definition is inspired by, and encompasses, several combinatorial/geometric constructions that have been applied to study local cohomology and connectedness properties of spectra. Given a positive integer ℓ, we use $[\ell]$ to denote the set $\{1, \ldots, \ell\}$.

Definition 2.1 (Codimension simplicial complexes)**.** Suppose that a ring S has dimension d. If S has ℓ minimal primes, fix an ordering $P_1, \ldots, P_\ell$ of them. Given an integer $t \geq 0$, the *t-th codimension complex* is the simplicial complex $\Lambda_t(S) \subseteq 2^{[\ell]}$ defined as follows: $\{i_0, \ldots, i_j\} \subseteq [\ell]$ is a face of $\Lambda_t(S)$ precisely if

$$\dim(S/(P_{i_0} + \ldots + P_{i_j})) \geq d - t.$$

i.e., the dimension of the intersection of the irreducible components $\mathbb{V}(P_{i_0}), \ldots, \mathbb{V}(P_{i_j})$ of $\operatorname{Spec}(S)$ is at least $d - t$.

In addition, we define $\Lambda_{-1}(S)$ to be the empty simplicial complex $\{\varnothing\}$.

Given integers $-1 \leq s < t$, observe that $\Lambda_s(S)$ is a subcomplex of $\Lambda_t(S)$ that may have fewer vertices; it is possible that $\Lambda_t(S)$ is disconnected, while $\Lambda_s(S)$ is connected (and vice versa). The complex $\Lambda_0(S)$ consists only of vertices: those corresponding to the minimal primes P_i for which $\dim(S/P_i) = \dim S$. Hence S is equidimensional if and only if $\Lambda_0(S) = \{\emptyset, \{1\}, \ldots, \{\ell\}\}$, which happens if and only if for every $t \geq 0$, $\Lambda_t(S)$ has vertex set $[\ell]$.

In general, if $t \geq d-1$, $\Lambda_t(S)$ must have vertex set $[\ell]$, and if $t \geq d$, then $\Lambda_t(S)$ is the full simplex on $[\ell]$. Hence

$$\{\varnothing\} = \Lambda_{-1}(S) \subsetneq \Lambda_0(S) \subseteq \Lambda_1(S) \subseteq \cdots \subseteq \Lambda_d(S) = 2^{[\ell]}$$

is a filtration of the full simplex on ℓ vertices.

The following example will be revisited throughout the paper.

Example 2.2. Let $R = \Bbbk[x_1, \ldots, x_6]$, where $\Bbbk$ is a field, and let I be the intersection of the ideals $I_1 = \langle x_1, x_2 \rangle$, $I_2 = \langle x_3, x_4 \rangle$, $I_3 = \langle x_5, x_6 \rangle$, and $I_4 = \langle x_1, x_6 \rangle$ of R, and let P_i denote the image of I_i in $S = R/I$. Then S has dimension four, $\Lambda_{-1}(S) = \{\varnothing\}$ as always, and under the ordering P_1, P_2, P_3, P_4, $\Lambda_t(S)$, for $0 \leq t \leq 4$, appear in Figure 2.2.1. The complex $\Lambda_4(S)$ is the full 3-simplex, and $\Lambda_3(S)$ is its boundary, removing $\{1, 2, 3\}$.

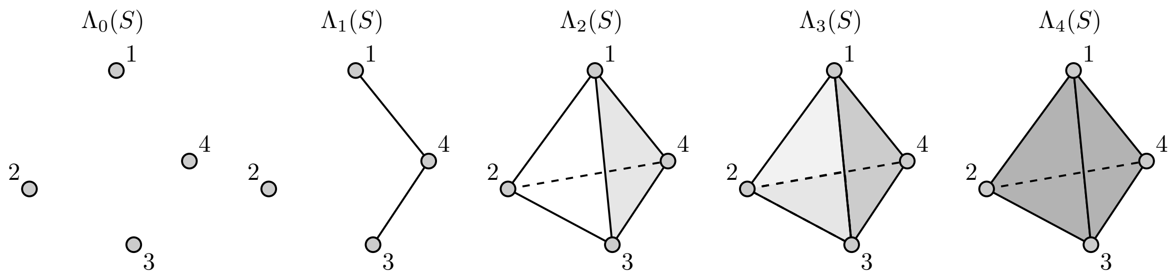

FIGURE 2.2.1. The codimension complexes of S from Example 2.2.

The codimension complexes are closely related to other combinatorial tools that have been used to study local cohomology. Consider the following condition on a d-dimensional ring S.

(DIM) For all ideals J of S, $\text{height}_S(J) + \dim(S/J) = d$.

Notice that if S satisfies (DIM), it must be equidimensional since if P, Q are minimal primes of S, then the condition forces $\dim(S/P) = \dim(S/Q) = d$. The condition (DIM) is satisfied by, for instance, any complete, equidimensional local ring containing a field, any finitely-generated algebra over a field that is a domain [Eis95, Corollary 13.4], and by any Cohen-Macaulay local ring [Aut, Lemma 10.104.4].

Recall that the *Hochster-Huneke graph* of S, also called *(Hartshorne's) dual graph*, has vertices indexed by the minimal primes P of S for which $\dim(S/P) = \dim(S)$, and there is an edge between vertices corresponding to P and Q if and only if $\text{height}_S(P + Q) = 1$ [HH94, Definition 3.4]; its connectedness is implicitly used earlier in [Har62]. This graph has been used to characterize the so-called "highest" Lyubeznik number [Lyu06, Zha07]; see also [HNPW19].

If S satisfies (DIM), then its Hochster-Huneke graph is the 1-skeleton of $\Lambda_1(S)$, but this is not true in general. However, the 0-skeleton of the Hochster-Huneke graph always coincides with $\Lambda_0(S)$. To illustrate the fact that these constructions can differ if (DIM) fails, let S be the quotient of $\Bbbk[x_1, x_2, x_3, x_4]$ modulo the intersection of $I_1 = \langle x_1 \rangle$, $I_2 = \langle x_2 \rangle$, and $I_3 = \langle x_3, x_4\rangle$, and for $1 \le i \le 3$. Under the ordering P_1, P_2, P_3 of the minimal primes of S, where P_i denotes the image of I_i in S, the Hochster-Huneke graph of S consists of vertices 1 and 2 and the edge between them. On the other hand, $\Lambda_0(S)$ consists only of the vertices 1 and 2, and $\Lambda_1(S)$ contains vertices 1, 2, and 3, and an edge between 1 and 2.

For $1 \le t \le d-1$, if S satisfies (DIM), then the 1-skeleton of $\Lambda_t(S)$ is the graph $\Gamma_t(S)$ used by the second author, Núñez-Betancourt, and Spiroff to characterize, in terms of local cohomology, when the connectedness dimension of the spectrum of a ring equals 1 [NSW19, Definition 2.2]. Hence $\Lambda_t(S)$ is connected precisely when $\Gamma_t(S)$ is connected, and in general, these objects have the same number of connected components.

Moreover, when S is local and satisfies (DIM), $\Lambda_{d-1}(S)$ is a complex used by Lyubeznik to study cohomological dimension [Lyu07, Theorem 1.1]. This complex is later called upon in results he obtained with Katzman and Zhang [KLZ16, Definition 1.1].

Remark 2.3. In the setting of Definition 2.1, suppose that $S \cong R/I$, for some ideal I of an n-dimensional ring R that satisfies (DIM). Suppose that $P_1, \dots, P_\ell \in \operatorname{Spec}(S)$ correspond to $I_1, \dots, I_\ell \in \operatorname{Spec}(R)$, respectively, modulo I. Given $\sigma = \{i_0, \dots, i_j\} \subseteq [\ell]$, then $\sigma \in \Lambda_t(S)$ if and only if $\dim(R/(I_{i_0} + \cdots + I_{i_j})) \ge d - t$ since $S/(P_{i_0} + \cdots + P_{i_j}) \cong R/(I_{i_0} + \cdots + I_{i_j})$. Let $J = I_{i_0} + \cdots + I_{i_j}$. Since R satisfies (DIM), this holds if and only if $\operatorname{height}_R(J) = n - \dim(R/J) \le n - (d-t)$, i.e., $t \ge \operatorname{height}_R(J) - \operatorname{height}_R(I)$. As a consequence, $\sigma \in \Lambda_t(S) \setminus \Lambda_{t-1}(S)$ if and only if $t = \operatorname{height}_R(J) - \operatorname{height}_R(I)$.

We use standard notation in dealing with simplicial homology. Suppose that $\Delta \subseteq 2^{[\ell]}$ is a simplicial complex and $j \in \mathbb{Z}$. Given a ring A, $C_j(\Delta; A)$ is the free A-module with basis consisting of the j-simplicies of Δ, i.e., the elements $\sigma \in \Delta$ such that $|\sigma| = j+1$. The j-th simplicial homology module of Δ with coefficients in A, denoted $H_j(\Delta; A)$, is the j-th homology of the complex

$$\cdots \to C_k(\Delta; A) \xrightarrow{\partial_k} C_{k-1}(\Delta; A) \to \cdots \to C_1(\Delta; A) \xrightarrow{\partial_1} C_0(\Delta; A) \to 0$$

such that, if $\sigma = \{i_0, \dots, i_j\}$, where $0 \le i_0 < i_1 < \cdots < i_j \le \ell$, then $\partial_j(\sigma) = \sum_{k=0}^{j} (-1)^k \{i_0, \dots, \widehat{i_k}, \dots, i_j\}$. The j-th reduced homology of Δ with coefficients in A, denoted $\widetilde{H}_j(\Delta; A)$, is the j-th homology of the augmented complex $\widetilde{C}_\bullet(\Delta; A)$ defined as

$$\cdots \to C_k(\Delta; A) \xrightarrow{\partial_k} C_{k-1}(\Delta; A) \to \cdots \to C_1(\Delta; A) \xrightarrow{\partial_1} C_0(\Delta; A) \to A \to 0$$

where the last nonzero term is the rank 1 free A-module with basis $\varnothing$, the unique -1-simplex of Δ, and the map $C_0(\Delta; A) \to A$ sends each 0-simplex of Δ to $\varnothing$.

Given an arbitrary A-module M, $C_j(\Delta; M)$ is defined as $C_j(\Delta; A) \otimes_A M$. We often identify $C_j(\Delta; M)$ with the direct sum of M over the index set of the j-simplices of Δ, so that for a j-simplex σ and $u \in M$, $\sigma \otimes u \in C_j(\Delta; M)$ is written as σu, indicating that u appears in the copy of M at index σ. By abuse of notation, we use ∂_j to denote the naturally-induced map $C_j(\Delta; M) \to C_{j-1}(\Delta; M)$. These homomorphisms make $C_\bullet(\Delta; M)$ a complex whose j-th homology module is $H_j(\Delta; M)$, the j-th simplicial homology of Δ with coefficients in M. The reduced complex $\widetilde{C}_\bullet(\Delta, M)$ is defined as $\widetilde{C}_\bullet(\Delta; A) \otimes_A M$, and its j-th reduced homology module $\widetilde{H}_j(\Delta; M)$ of Δ with coefficients in M is its j-homology.

Suppose that Δ' is a subcomplex of Δ. Then the A-module $C_j(\Delta, \Delta'; M)$ is defined as the quotient $C_j(\Delta; M)/C_j(\Delta'; M)$, and $C_\bullet(\Delta, \Delta'; M)$ is again a complex with maps induced by the homomorphisms $\partial_j : C_j(\Delta; M) \to C_{j-1}(\Delta; M)$. The j-th homology of this complex is $H_j(\Delta, \Delta'; M)$, the j-th homology of Δ relative to Δ', which we also call the j-th relative homology of $\Delta' \subseteq \Delta$. Observe that under a similar construction, where $\widetilde{C}_\bullet(\Delta, \Delta'; M)$ is the complex whose j-th term is $\widetilde{C}_j(\Delta; M)/\widetilde{C}_j(\Delta'; M)$, we find that all reduced relative homology agrees with relative homology since $\widetilde{C}_{-1}(\Delta, \Delta'; M) = 0$.

We often make use of the long exact sequence in reduced simplicial homology with respect to $\Delta' \subseteq \Delta$:

$$\cdots \to \widetilde{H}_j(\Delta'; M) \to \widetilde{H}_j(\Delta; M) \to H_j(\Delta, \Delta'; M) \to \widetilde{H}_{j-1}(\Delta'; M) \to \cdots . \tag{2.1}$$

Remark 2.4. Given a d-dimensional ring S, since $\Lambda_d(S)$ is the full simplex on the vertices of S, it is homeomorphic to a point, so all its reduced homology vanishes. Hence for any module M over a ring A, the long exact sequence (2.1) with respect to $\Lambda_{d-1}(S) \subseteq \Lambda_d(S)$ yields, for all integers j, an A-module isomorphism

$$H_j(\Lambda_d(S), \Lambda_{d-1}(S); M) \cong \widetilde{H}_{j-1}(\Lambda_{d-1}(S); M).$$

In the following statement, recall that valid indices for the codimension complexes are integers $t \geq -1$, so can only have $i \leq d+1$ in the set of which we are taking the minimum.

Lemma 2.5. *Given rings S and A and a nonzero A-module M, $\widetilde{H}_{-1}(\Lambda_{-1}(S); M) \cong M$, and if $d = \dim S$, then*

$$0 \leq \min\{i + j \mid \widetilde{H}_j(\Lambda_{d-i}(S); M) \neq 0\} \leq d.$$

Proof. The first statement follows from the fact that $\Lambda_{-1}(S)$ is defined as $\{\varnothing\}$, so that $\widetilde{C}_\bullet(\Lambda_{-1}(S); M)$ has only one nonzero term, M, at index -1. Hence $\widetilde{H}_j(\Lambda_{-1}(S); M)$ is zero unless $j = -1$, in which case it is isomorphic to M. As a consequence, the minimum is well-defined and at most $(d + 1) + (-1) = d$.

For the remaining inequality, we argue that if $\widetilde{H}_j(\Lambda_{d-i}(S); M) \neq 0$, then $i + j \geq 0$. We have already addressed the case that $d - i = -1$, i.e., $i = d + 1$, so we can assume that $\Lambda_{d-i}(S)$ contains at least one 0-cell, so $\widetilde{C}_0(\Lambda_{d-i}(S); M)$ maps onto $\widetilde{C}_{-1}(\Lambda_{d-i}(S); M) \cong M$, and $\widetilde{H}_j(\Lambda_{d-i}(S); M) = 0$ for $j < 0$. Hence if $\widetilde{H}_j(\Lambda_{d-i}(S); M) \neq 0$, then $j \geq 0$. If $i < 0$, then $\Lambda_{d-i}(S) = \Lambda_d(S)$ is homeomorphic to a point, so $\widetilde{H}_j(\Lambda_{d-i}; M) = 0$ for all $j \in \mathbb{Z}$. Hence $i \geq 0$, so that $i + j \geq 0$. $\square$

Proposition 2.6. *Given rings S and A and a nonzero A-module M, if $d = \dim S$, then*

$$\min\{i + j \mid \widetilde{H}_j(\Lambda_{d-i}(S); M) \neq 0\} = \min\{i + j \mid H_j(\Lambda_{d-i}(S), \Lambda_{d-i-1}(S); M) \neq 0\}.$$

Proof. Let $\Lambda_t = \Lambda_t(S)$. The left-hand minimum, which we will call α, is well defined by Lemma 2.5, so we can fix $i, j \in \mathbb{Z}$ such that $\widetilde{H}_j(\Lambda_{d-i}; M) \neq 0$ and $i + j = \alpha$, i.e., $j = \alpha - i$. Then $\gamma = \min\{i \mid \widetilde{H}_{\alpha-i}(\Lambda_{d-i}; M) \neq 0\}$ is positive since all reduced homology of Λ_d vanishes.

By the definition of α, $\widetilde{H}_{\alpha-i-1}(\Lambda_{d-i}; M) = 0$ for every integer i, so a portion of (2.1) with respect to $\Lambda_{d-i-1} \subseteq \Lambda_{d-i}$ has the form

$$\cdots \to \widetilde{H}_{\alpha-i}(\Lambda_{d-i}; M) \to H_{\alpha-i}(\Lambda_{d-i}, \Lambda_{d-i-1}; M) \to \widetilde{H}_{\alpha-i-1}(\Lambda_{d-i-1}; M) \to 0.$$

For any $i < \gamma$, $\widetilde{H}_{\alpha-i}(\Lambda_{d-i}; M) = 0$, so that $H_{\alpha-i}(\Lambda_{d-i}, \Lambda_{d-i-1}; M) \cong \widetilde{H}_{\alpha-i-1}(\Lambda_{d-i-1}; M)$. If $i < \gamma - 1$, this further implies that $H_{\alpha-i}(\Lambda_{d-i}, \Lambda_{d-i-1}; M) = 0$ since $\widetilde{H}_{\alpha-i-1}(\Lambda_{d-i-1}; M) =$

0 by our choice of γ. However, if $i = \gamma - 1$, we see that $H_{\alpha-\gamma+1}(\Lambda_{d-\gamma+1}, \Lambda_{d-\gamma}; M) \cong \widetilde{H}_{\alpha-\gamma}(\Lambda_{d-\gamma}; M) \neq 0$. Hence the right-hand minimum, which we will call β, is well defined, and $\beta \leq (\alpha - \gamma + 1) + (\gamma - 1) = \alpha$.

Now let i, j be integers such that $i + j < \alpha$. Then $\widetilde{H}_j(\Lambda_{d-i}; M) = \widetilde{H}_{j-1}(\Lambda_{d-i-1}; M) = 0$, and the portion of long exact sequence in relative homology (2.1) is of the form

$$0 = \widetilde{H}_j(\Lambda_{d-i}; M) \to H_j(\Lambda_{d-i}, \Lambda_{d-i-1}; M) \to \widetilde{H}_{j-1}(\Lambda_{d-i-1}; M) = 0,$$

which forces $H_j(\Lambda_{d-i}, \Lambda_{d-i-1}; M) = 0$. Hence $\beta \geq \alpha$ as well. □

3. Relative Simplicial Homology of Consecutive Codimension Complexes

We shift our focus to central linear subspace arrangements. An ideal of a polynomial ring over a field defines a *central linear subspace arrangement* if its minimal primes are generated by linear forms. In particular, a squarefree monomial ideal defines a central linear subspace arrangement.

Remark 3.1. Let $R = \Bbbk[x_1, \ldots, x_n]$, where $\Bbbk$ is a field, and let J be an ideal generated by linear forms; for instance, J could be an arbitrary sum of the I_i in Setup 3.3. Then J is a $\Bbbk$-vector subspace of the n-dimensional $\Bbbk$-vector space of all linear forms in R, so J has a basis consisting of linear forms $g_1, \ldots, g_t$, for some $t \leq n$. Then the map sending $x_i \mapsto g_i$ for $1 \leq i \leq n$, fixing $x_{m+1}, \ldots, x_n$, defines a linear change of coordinates on R.

In particular, J is a prime ideal of R of height t, and R/J is isomorphic to a polynomial ring over $\Bbbk$ of dimension $n-t$. Moreover, this change of coordinates induces an isomorphism of the Čech-like complexes defining local cohomology with support in J and that with support in $\langle x_1, \ldots, x_t \rangle$, where t is the height of J in R. Hence $H^i_J(R) \cong H^i_{\langle x_1,\ldots,x_t\rangle}(R)$ for all i, and in particular, $H^i_J(R) \neq 0$ if and only if $i = \operatorname{height}_R J = t$.

Remark 3.2. If P and Q are prime ideals of a ring R, observe that $\dim R/(P+Q) = \dim R/P$ if and only if $Q \subseteq P$. Hence if $P_1, \ldots, P_\ell$ are prime ideals of R such that $P_1 + \cdots + P_j$ is itself prime for any $1 < j \leq \ell$, then $\dim R/(P_1 + \cdots + P_j) = \dim R/(P_1 + \cdots + P_{j-1})$ if and only if $P_j \subseteq P_1 + \cdots + P_{j-1}$, i.e., if and only if $P_1 + \cdots + P_{j-1} = P_1 + \cdots + P_j$. In particular, this holds if R is a polynomial ring over a field and the P_i are generated by linear forms (see Remark 3.1).

This is not necessarily the case for arbitrary primes: Consider the principal ideals P_1, P_2, and P_3 of $\Bbbk[x, y]$ generated by $x, x + y^2$, and y, respectively. Then

$$\dim R/(P_1 + P_2) = \dim \Bbbk[y]/\langle y^2\rangle = 0 = \dim \Bbbk = \dim R/(P_1 + P_2 + P_3),$$

while $P_3 = \langle y\rangle \subsetneq \langle x, y^2\rangle = P_1 + P_2$.

We will often adopt the following setting in the remainder of the paper.

Setup 3.3. Let $R = \Bbbk[x_1, \ldots, x_n]$, where $\Bbbk$ is a field, and let $\mathfrak{m}$ denote its homogeneous maximal ideal. Let I be an ideal of R of height h that defines a central linear subspace arrangement. Moreover, let $I_1, \ldots, I_\ell$ denote the minimal primes of I. If $d = n - h$, this means that $S = R/I$ has dimension d. Given an integer $t \geq -1$, let $\Lambda_t(S)$ denote the t-th codimension complex of S (see Definition 2.1) with respect to the ordering $\overline{I}_1, \ldots, \overline{I}_\ell$ of its minimal primes, where $\overline{I}_i$ denotes the image of I_i modulo I.

The image of the following function consists of prime ideals generated by linear forms.

Definition 3.4. Under Setup 3.3, let $\mathbb{J} : 2^{[\ell]} \to \operatorname{Spec}(R)$ denote the function sending $\sigma = \{i_0, \ldots, i_j\}$ to

$$\mathbb{J}(\sigma) = I_{i_0} + \cdots + I_{i_j} \subseteq R.$$

Definition 3.5. Adopt Setup 3.3, and let M be a module over a ring A. Suppose that J is in the image of $\Lambda_t(S)$ under $\mathbb{J}$ for some integer $t \geq -1$. For any integer k, let $C_k^J(\Lambda_t(S); A)$ denote the free A-submodule of $C_k(\Lambda_t(S); A)$ with basis consisting of the k-simplices σ of $\Lambda_t(S)$ such that $\mathbb{J}(\sigma) = J$, and let $C_k^J(\Lambda_t(S); M)$ denote the corresponding submodule of $C_k(\Lambda_t(S); M)$. Let ∂_k^J denote the restriction of the homomorphism $\partial_k : C_k(\Lambda_t(S); M) \to C_{k-1}(\Lambda_t(S); M)$ to $C_k^J(\Lambda_t(S); M)$. Finally, let $C_k^J(\Lambda_t(S), \Lambda_{t-1}(S); M) = C_k^J(\Lambda_t(S); M)/C_k^J(\Lambda_{t-1}(S); M)$.

For any integers $t \geq -1$ and $k \geq 0$, $C_k(\Lambda_{t-1}(S); M)$ is naturally an A-submodule of $C_k(\Lambda_t(S); M)$, the direct sum of copies of M over index set of k-simplices in $\Lambda_t(S)$ that are also in $\Lambda_{t-1}(S)$. With this identification, we have the following.

Lemma 3.6. *Under Setup 3.3, fix an integer $t \geq -1$ and an ideal $J \in \mathbb{J}(\Lambda_t(S))$. For any integer $k \geq 0$, the image of ∂_k^J is contained in the submodule $C_{k-1}^J(\Lambda_t(S); M) + C_{k-1}(\Lambda_{t-1}(S); M)$ of $C_{k-1}(\Lambda_t(S); M)$.*

Proof. Consider $\sigma = \{i_0, \ldots, i_k\} \in \Lambda_t(S)$, where $1 \leq i_0 < \cdots < i_k \leq \ell$. If $\mathbb{J}(\sigma) = J$, then for $u \in M$, $\partial_k(\sigma u) = \sum (-1)^j \sigma_j u$, where the sum ranges over $0 \leq j \leq k$, and $\sigma_j = \{i_0, \ldots, \widehat{i_j}, \ldots, i_k\}$. Then $I_{i_j} \subseteq I_{i_1} + \cdots + \widehat{I_{i_j}} + \cdots + I_{i_k} = \mathbb{J}(\sigma_j)$ if and only if $\mathbb{J}(\sigma_j) = J$, which further holds if and only if $\sigma_j u \in C_{k-1}^J(\Lambda_t(S); M)$ (see Remark 3.2). Otherwise, $\dim(R/\mathbb{J}(\sigma_j)) > \dim(R/\mathbb{J}(\sigma)) = d - t$, so noting Remark 2.3, $\sigma_j u \in C_{k-1}(\Lambda_{t-1}(S); M)$. $\square$

Remark 3.7. Given integers t and k, observe that for any $J \in \mathbb{J}(\Lambda_t(S))$,

$$C_k(\Lambda_{t-1}(S); M) \cap C_k^J(\Lambda_t(S); M) = C_k^J(\Lambda_{t-1}(S); M).$$

Indeed, an element of the left-hand side is a sum of elements of $C_k(\Lambda_t(S); M)$ of the form σu, where $u \in M$ and $\sigma \in \Lambda_{t-1}(S)$ is a k-simplex such that $\mathbb{J}(\sigma) = J$.

Theorem 3.8. *Under Setup 3.3, fix an integer $t \geq -1$. For any ideal $J \in \mathbb{J}(\Lambda_t(S))$, $C_\bullet^J(\Lambda_t(S), \Lambda_{t-1}(S); M)$ forms a complex of A-modules with maps induced by the ∂_k^J. Moreover, if $\Omega = \Lambda_t(S)$ or $\Omega = \Lambda_t(S) \setminus \Lambda_{t-1}(S)$, there is an isomorphism of complexes*

$$C_\bullet(\Lambda_t(S), \Lambda_{t-1}(S); M) \cong \bigoplus_{J \in \mathbb{J}(\Omega)} C_\bullet^J(\Lambda_t(S), \Lambda_{t-1}(S); M).$$

As a consequence, $H_k(\Lambda_t(S), \Lambda_{t-1}(S); M) \cong \bigoplus_{J \in \mathbb{J}(\Omega)} H_k(C_\bullet^J(\Lambda_t(S), \Lambda_{t-1}(S); M))$.

Proof. First we justify that for $k \in \mathbb{Z}$, ∂_k^J induces a differential from $C_k^J(\Lambda_t(S), \Lambda_{t-1}(S); M)$ to $C_{k-1}^J(\Lambda_t(S), \Lambda_{t-1}(S); M)$. Consider the composition

$$C_k^J(\Lambda_t(S); M) \xrightarrow{\partial_k^J} C_{k-1}(\Lambda_t(S); M) \twoheadrightarrow C_{k-1}(\Lambda_t(S), \Lambda_{t-1}(S); M),$$

where the second map is the natural quotient modulo $C_{k-1}(\Lambda_{t-1}; M)$. Lemma 3.6 tells us that the image of the composition sits inside the submodule

$$C_{k-1}^J(\Lambda_t(S); M) / \left(C_{k-1}(\Lambda_{t-1}(S); M) \cap C_{k-1}^J(\Lambda_t(S); M)\right),$$

of $C_{k-1}(\Lambda_t(S), \Lambda_{t-1}(S); M) = C_{k-1}(\Lambda_t(S); M)/C_{k-1}(\Lambda_{t-1}(S); M)$, which, by Remark 3.7, is precisely $C^J_{k-1}(\Lambda_t(S), \Lambda_{t-1}(S); M)$. As $C^J_k(\Lambda_{t-1}(S); M)$ is in its kernel, this induces the desired map.

These homomorphisms make $C^J_\bullet(\Lambda_t(S), \Lambda_{t-1}(S); M)$ a complex since any $\partial_{k-1} \circ \partial_k = 0$, so a composition of two consecutive induced maps certainly vanishes.

The decomposition of $C_\bullet(\Lambda_t(S), \Lambda_{t-1}(S); M)$ now follows from the fact that the k-simplices of $\Lambda_t(S)$ is the disjoint union of those with image J under the function $\mathbb{J}$, as $J \in \operatorname{Spec}(R)$ varies over the image of $\Lambda_t(S)$ under $\mathbb{J}$, after noticing that if J is in the image of $\Lambda_{t-1}(S)$, then $C^J_\bullet(\Lambda_t(S), \Lambda_{t-1}(S); M)$ is the zero complex. In particular, this observation justifies that both options for Ω are valid. The final statement now follows by taking homology of either side of the decomposition of $C_\bullet(\Lambda_t(S), \Lambda_{t-1}(S); M)$. □

Example 3.9. Returning to our running example, Example 2.2, observe that

$$\Lambda_2(S) \setminus \Lambda_1(S) = \{\{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{1,3,4\}\}.$$

Let M be a module over a ring A. Then the nonzero terms of the complex $C_\bullet = C_\bullet(\Lambda_2(S), \Lambda_1(S); M)$ are $C_1 \cong M^{\oplus 4}$ and $C_2 \cong M$, and the only nonzero map is ∂_2, which sends, for $u \in M$, $\{1,3,4\}u \mapsto \{1,3\}u$. Now, $\mathbb{J}(\{1,2\}) = \langle x_1, x_2, x_3, x_4 \rangle$, $\mathbb{J}(\{2,3\}) = \langle x_3, x_4, x_5, x_6 \rangle$, and $\mathbb{J}(\{2,4\}) = \langle x_1, x_3, x_4, x_6 \rangle$, while

$$\mathbb{J}(\{1,3\}) = \mathbb{J}(\{1,3,4\}) = \langle x_1, x_2, x_5, x_6 \rangle.$$

Hence by Theorem 3.8, $C_\bullet$ is the direct sum of the complexes $C^J_\bullet = C^J_\bullet(\Lambda_2(S), \Lambda_1(S); M)$ appearing in Figure 3.9.1, where, for $\sigma \in \Lambda_2(S)$, $M\sigma$ denotes the copy of M at index σ. Since ∂_2 is an isomorphism, by Theorem 3.8, the only nonzero homology module of $C_\bullet$ is

$$H_1(C_\bullet) = M\{1,2\} \oplus M\{2,3\} \oplus M\{2,4\}.$$

J	$0 \longrightarrow C^J_2 \longrightarrow C^J_1 \longrightarrow 0$
$\langle x_1, x_2, x_3, x_4 \rangle$	$0 \longrightarrow 0 \longrightarrow M\{1,2\} \longrightarrow 0$
$\langle x_3, x_4, x_5, x_6 \rangle$	$0 \longrightarrow 0 \longrightarrow M\{2,3\} \longrightarrow 0$
$\langle x_1, x_3, x_4, x_6 \rangle$	$0 \longrightarrow 0 \longrightarrow M\{2,4\} \longrightarrow 0$
$\langle x_1, x_2, x_5, x_6 \rangle$	$0 \longrightarrow M\{1,3,4\} \longrightarrow M\{1,3\} \longrightarrow 0$

FIGURE 3.9.1. The components of $C_\bullet(\Lambda_2(S), \Lambda_1(S); M)$ from Example 3.9.

4. The Mayer-Vietoris Spectral Sequence in Local Cohomology with Support in an Ideal Defining a Linear Subspace Arrangement

The following spectral sequence was introduced by Àlvarez Montaner, García López, and Zarzuela Armengou [ÀGZ03] in the case that $I_1, \dots, I_\ell$ are ideals in a polynomial ring R over a field that define the irreducible components of a subspace arrangement, and the definition was extended to an arbitrary intersections of an ideal ideal I of a Noetherian ring as an intersection by Lyubeznik [Lyu07, Theorem 2.1 and Remark 2.2].

Definition/Theorem 4.1 (Mayer-Vietoris spectral sequence in local cohomology)**.** Given ideals $I_1, \dots, I_\ell$ of a Noetherian ring R, let $I = I_1 \cap \dots \cap I_\ell$. There is a spectral sequence

$$E_1^{-p,q} = \bigoplus_{1 \le i_0 < \cdots < i_p \le \ell} H^q_{I_{i_0}+\cdots+I_{i_p}}(R) \Rightarrow H^{q-p}_I(R) \tag{4.1}$$

such that the differential $\phi_1^{-p,q} : E_1^{-p,q} \to E_1^{-p+1,q}$ on $H^q_J(R)$, where $J = I_{i_0} + \cdots + I_{i_p}$, is defined as $\sum_{k=0}^{p}(-1)^k h^q_{J,J_k}$, where $J_k = I_{i_0} + \cdots + \widehat{I}_{i_k} + \cdots + I_{i_p}$ and $h^q_{J,J_k} : H^q_J(R) \to H^q_{J_k}(R)$ is the natural map induced by the inclusion $J_k \subseteq J$.

Let $I_1, \dots, I_\ell$ denote the minimal primes of an ideal I of a Noetherian ring R. Then since $I_1 \cap \dots \cap I_\ell = \sqrt{I}$ and local cohomology with support in I depends only on its radical, note that one has the spectral sequence (4.1) regardless of whether I is radical, i.e., whether $I = I_1 \cap \dots \cap I_\ell$.

Remark 4.2. Under Setup 3.3, as an arbitrary sum of the I_i is again generated by linear forms, the local cohomology of R with support in such a sum is only nonzero at the index equal to its height (see Remark 3.1). Hence in the notation of Definition/Theorem 4.1, h^q_{J,J_k} is nonzero if and only if $J_k = J$, i.e., when $I_{i_k} \subseteq J_k$ (see Remark 3.2), and in this case it is, up to a sign $(-1)^k$, the identity map. Hence for $u \in H^q_J(R)$,

$$\phi_1^{-p,q}(u) = \sum_{0 \le k \le p,\, J_k = J} (-1)^k \operatorname{id}_{H^q_J(R)}(u).$$

Proposition 4.3. *Under Setup 3.3, the terms of the Mayer-Vietoris spectral sequence in local cohomology with respect to $I_1, \dots, I_\ell$ have the form, for $p, q \in \mathbb{Z}$,*

$$E_1^{-p,q} = \bigoplus_{\substack{\sigma \in \Lambda_{q-h}(S) \\ |\sigma| = p+1}} H^q_{\mathbb{J}(\sigma)}(R).$$

Moreover, $H^q_{\mathbb{J}(\sigma)}(R) \neq 0$ if and only if $\sigma \notin \Lambda_{q-h-1}(S)$.

Proof. Consider a summand of $E_1^{-p,q}$ from (4.1), that is, $M = H^q_{I_{i_0}+\cdots+I_{i_p}}(R)$ for some $1 \le i_0 < \cdots < i_p \le \ell$. Then $M \neq 0$ if and only if $\operatorname{height}_R(I_{i_0} + \cdots + I_{i_p}) = q$ (see Remark 3.1), i.e., precisely if

$$\sigma = \{i_0, \dots, i_p\} \in \Lambda_{q-h}(S) \setminus \Lambda_{q-h-1}(S)$$

by Remark 2.3. The statement now follows from the definition of the function $\mathbb{J}$. □

Observe that in the context of the Mayer-Vietoris spectral sequence for local cohomology, for any $q \in \mathbb{Z}$, $G_\bullet = E_1^{-\bullet,q}$ is a homological complex of R-modules, with, for $p \in \mathbb{Z}$, differential $\phi_1^{-p,q}$ from $G_p = E_1^{-p,q}$ to $G_{p-1} = E_1^{-(p-1),q}$.

Theorem 4.4. *Adopt Setup 3.3, and consider the Mayer-Vietoris spectral sequence in local cohomology with respect to $I_1, \dots, I_\ell$. Given any integer q, there is a natural isomorphism of (homological) complexes of R-modules*

$$E_1^{-\bullet,q} \cong \bigoplus_{J \in \mathbb{J}(\Omega)} C^J_\bullet(\Lambda_{q-h}(S), \Lambda_{q-h-1}(S); H^q_J(R))$$

where Ω can be taken to be either $\Lambda_{q-h}(S)$ or $\Lambda_{q-h}(S) \setminus \Lambda_{q-h-1}(S)$.

Proof. First observe that for any integer p, the p-th terms of the two complexes are isomorphic: By Proposition 4.3, $E_1^{-p,q}$ is the direct sum of local cohomology modules $H^q_{\mathbb{J}(\sigma)}(R)$, where σ ranges over all p-simplices of $\Lambda_{q-h}(S)$ that are not in $\Lambda_{q-h-1}(S)$. Given $J \in \mathbb{J}(\Lambda_{q-h}(S) \setminus \Lambda_{q-h-1}(S))$, observe that for $p \in \mathbb{Z}$, $C^J_p(\Lambda_{q-h}(S), \Lambda_{q-h-1}(S); H^q_J(R))$ can be canonically identified with the direct sum of copies of $H^q_J(R)$ over index set consisting of all p-simplices $\sigma \in \Lambda_{q-h}(S)$ for which $\mathbb{J}(\sigma) = J$, or over the subset of these not in $\Lambda_{q-h-1}(S)$.

As all such p-simplices σ correspond to precisely one such J,

$$\bigoplus_{J \in \mathbb{J}(\Omega)} C^J_p(\Lambda_{q-h}(S), \Lambda_{q-h-1}(S); H^q_J(R))$$

is also this direct sum, arranged so that $C^J_p(\Lambda_{q-h}(S), \Lambda_{q-h-1}(S); H^q_J(R))$ contains all copies of $H^q_J(R)$, ranging over those σ in the preimage of J under $\mathbb{J}$.

Fix some $\sigma = \{i_0, \ldots, i_p\} \in \Lambda_{q-h}(S) \setminus \Lambda_{q-h-1}(S)$, where $i_0 < \cdots < i_p$. For $0 \le k \le p$, let $\sigma_k = \{i_0, \ldots, \widehat{i_k}, \ldots, i_p\}$. As pointed out in Remark 4.2, if $u \in H^q_{\mathbb{J}(\sigma)}(R)$, then

$$\phi_1^{-p,q}(u) = \sum_{0 \le k \le p,\, \mathbb{J}(\Lambda_k) = \mathbb{J}(\sigma)} (-1)^k \operatorname{id}_{H^q_{\mathbb{J}(\sigma)}(R)}(u).$$

In particular, the image of $H^q_{\mathbb{J}(\sigma)}(R)$ is inside the direct sum of local cohomology modules with support in the same ideal.

Now let $J = \mathbb{J}(\sigma)$, and consider $u \neq 0$ in the corresponding copy of $H^q_J(R)$ in the module $C^J_p(\Lambda_{q-h}(S), \Lambda_{q-h-1}(S); H^q_J(R))$. Then the image under ∂^J_p of σu is $\sum_{k=0}^{p} (-1)^k \sigma_k u$ modulo $C^J_{p-1}(\Lambda_{q-h-1}(S); H^q_J(R))$, and $\sigma_k u$ is not zero, i.e., $\sigma_k u \notin C^J_{p-1}(\Lambda_{q-h-1}(S); H^q_J(R))$, precisely if $\operatorname{height}_R(\mathbb{J}(\sigma_k)) = q$, which is true if and only if $\mathbb{J}(\sigma_k) = J = \mathbb{J}(\sigma)$ (see Remark 2.3). Hence the maps in the complexes are compatible with the natural identification of their terms, and the complexes are isomorphic. □

Remark 4.5 (Universal coefficient theorem). Given a principal ideal domain A, suppose that M is an A-module and $G_\bullet$ is a chain complex of free A-modules. The universal coefficient theorem states that for every integer k, there exists a short exact sequence of A-modules

$$0 \to H_k(G_\bullet; A) \otimes_A M \to H_k(G_\bullet; M) \to \operatorname{Tor}_1^A(H_{k-1}(G_\bullet; A), M) \to 0$$

that splits, though not naturally [Mil21, Theorem 24.1]. Hence if A is a field and M is a vector space over A, then the above Tor-module vanishes and for every $k \in \mathbb{Z}$,

$$H_k(G_\bullet; A) \otimes_A M \cong H_k(G_\bullet; M).$$

Remark 4.6. Under Setup 3.3, given an ideal J of R generated by linear forms, each local cohomology of R with support in J is naturally a $\Bbbk$-vector space. Indeed, by Remark 3.1, the only nonvanishing local cohomology module is at index $h = \operatorname{height}_R(I)$, and

$$H^h_J(R) \cong R_{x_1 \cdots x_q} \Big/ \sum_{1 \le i \le q} R_{x_1 \cdots \widehat{x_i} \cdots x_q} \cong \bigoplus_{\substack{i_1, \ldots, i_h \ge 1 \\ i_{h+1}, \ldots, i_n \ge 0}} \Bbbk \cdot x_1^{-i_1} \cdots x_h^{-i_h} x_{q+1}^{i_{h+1}} \cdots x_n^{i_n}.$$

Hence by the universal coefficient theorem (see Remark 4.5), for integers $t \ge 0$ and p, q,

$$H_p\left(C^J_\bullet(\Lambda_t(S), \Lambda_{t-1}(S); H^q_J(R)\right) \cong H_p\left(C^J_\bullet(\Lambda_t(S), \Lambda_{t-1}(S); \Bbbk\right) \otimes_\Bbbk H^q_J(R).$$

Corollary 4.7. *Adopt Setup 3.3, and consider the Mayer-Vietoris spectral sequence in local cohomology with respect to $I_1, \dots, I_\ell$. For all integers p, q, if $\Omega = \Lambda_{q-h}(S)$ or $\Omega = \Lambda_{q-h}(S) \setminus \Lambda_{q-h-1}(S)$, then the following hold.*

(a) *There is an R-module isomorphism*

$$E_2^{-p,q} \cong \bigoplus_{J \in \mathbb{J}(\Omega)} H_p\left(C^J_\bullet(\Lambda_{q-h}(S), \Lambda_{q-h-1}(S); H^q_J(R)\right).$$

(b) *There is a $\Bbbk$-vector space isomorphism*

$$E_2^{-p,q} \cong \bigoplus_{J \in \mathbb{J}(\Omega)} H_p\left(C^J_\bullet(\Lambda_{q-h}(S), \Lambda_{q-h-1}(S); \Bbbk\right) \otimes_\Bbbk H^q_J(R).$$

(c) *We have that $E_2^{-p,q} = 0$ if and only if $H_p(\Lambda_{q-h}(S), \Lambda_{q-h-1}(S); \Bbbk) = 0$.*

Proof. To see (a), take homology of both sides of the statement in Theorem 4.4; then apply the conclusion of Remark 4.6 to obtain (b). Toward (c), observe that if $\Omega = \Lambda_{q-h}(S) \setminus \Lambda_{q-h-1}(S)$, then for $\sigma \in \Omega$, $H^q_{\mathbb{J}(\sigma)}(R)$ is not zero (see Remarks 2.3 and 3.1). Hence the right-hand side of (b) vanishes if and only if $H^J_p(\Lambda_{q-h}(S), \Lambda_{q-h-1}(S); \Bbbk) = 0$ for all $J \in \mathbb{J}(\Omega)$, which happens if and only if $H_p(\Lambda_{q-h}(S), \Lambda_{q-h-1}(S); \Bbbk) = 0$ by Theorem 3.8. □

Example 4.8. We return to Example 2.2, and consider the Mayer-Vietoris spectral sequence with respect to I_1, I_2, I_3, and I_4. Given $1 \le i_1 < \cdots < i_j \le 6$, let $N_{i_1 i_2 \cdots i_j}$ denote $H^j_{\langle x_{i_1}, x_{i_2}, \dots, x_{i_j} \rangle}(R)$. The nonzero terms $E_1^{-p,q}$ appear in Figure 4.8.1; the two nonzero maps between such terms are indicated. Likewise, the nonzero terms $E_2^{-p,q}$ appear in Figure 4.8.2; there are no nonzero differentials on the E_2-page.

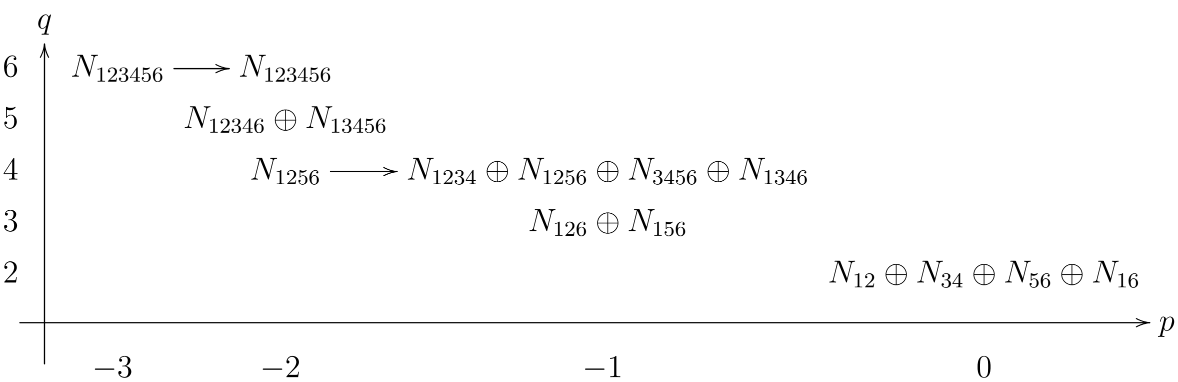

FIGURE 4.8.1. The nonzero terms $E_1^{-p,q}$, and nonzero differentials between them, from Example 4.8.

Remark 4.9. Under Setup 3.3, the Mayer-Vietoris spectral sequence with respect to $I_1, \dots, I_\ell$ degenerates at the E_2-page [ÀGZ03, Theorem 1.2]. Hence for every integer i,

$$E^i_\infty = \bigoplus_{p \ge 0} E_2^{-p,p+i} = \bigoplus_{p=0}^{n-i} E_2^{-p,p+i}$$

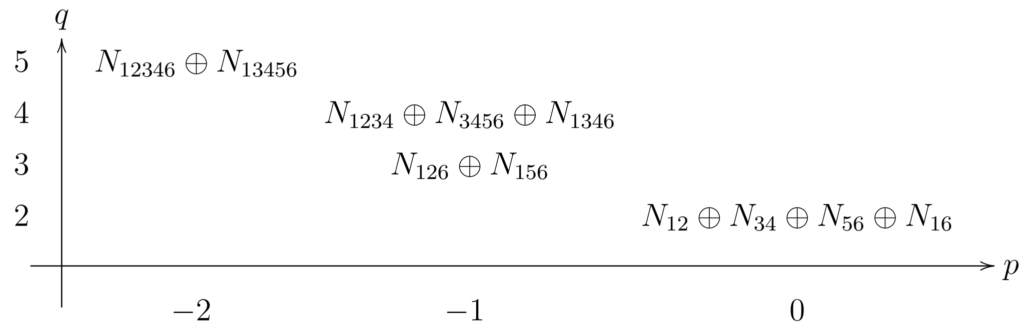

FIGURE 4.8.2. The nonzero terms $E_2^{-p,q}$ from Example 4.8.

is an associated graded module corresponding to a filtration of $H^i_I(R)$. Indeed, since any local cohomology module of R vanishes beyond n, the dimension of R, Corollary 4.7 implies that every $E_2^{-p,p+i} = 0$ for $p > n - i$, so one can restrict the index set to $0 \le p \le n - i$.

Corollary 4.10. *Adopt Setup 3.3, and consider the Mayer-Vietoris spectral sequence in local cohomology with respect to $I_1, \ldots, I_\ell$. For each integer i, there exists a filtration of R-modules*

$$0 = \mathcal{M}^i_{-1} \subseteq \mathcal{M}^i_0 \subseteq \mathcal{M}^i_1 \subseteq \cdots \subseteq \mathcal{M}^i_{n-i} = H^i_I(R)$$

such that for each $0 \le j \le n - i$,

$$\mathcal{M}^i_j / \mathcal{M}^i_{j-1} \cong E_2^{-j,i+j} \cong \bigoplus_{J \in \mathbb{J}(\Omega)} H_j\left(C^J_\bullet(\Lambda_{i+j-h}(S), \Lambda_{i+j-h-1}(S); H^{i+j}_J(R)\right)$$

where $\Omega = \Lambda_{i+j-h}(S)$ or $\Omega = \Lambda_{i+j-h}(S) \setminus \Lambda_{i+j-h-1}(S)$.

Proof. This follows from Corollary 4.7 and Remark 4.9. □

Observe that under Setup 3.3, for every integer q, the differential $\phi_1^{-1,q} : E_1^{-1,q} \to E_1^{0,q}$ is the zero map: By way of contradiction, suppose that $\phi_1^{-1,q}$ is nonzero on some summand $H^q_{\mathbb{J}(\sigma)}(R)$ of $E_1^{-1,q}$, so that $\sigma = \{i, j\}$ for some $1 \le i < j \le \ell$. Then $\phi_1^{-1,q}$ maps $H^q_{\mathbb{J}(\sigma)}(R) = H^q_{I_i+I_j}(R)$ into $H^q_{I_1}(R) \oplus H^q_{I_2}(R)$. Since the map is nonzero, q must be the height of one of the minimal primes I_1, I_2 of I. However, the height of $I_1 + I_2$ is greater than either height, forcing $H^q_{\mathbb{J}(\sigma)}(R) = 0$ by Remark 3.1. Hence for every integer q,

$$E_\infty^{0,q} = E_2^{0,q} = E_1^{0,q} = \bigoplus_{\text{height}_R(I_i) = q} H^q_{I_i}(R),$$

which, in the notation of Corollary 4.10, is (isomorphic to) the submodule $\mathcal{M}^q_0$ of $H^q_I(R)$.

5. LOCAL COHOMOLOGY WITH SUPPORT IN AN IDEAL DEFINING A CENTRAL LINEAR SUBSPACE ARRANGEMENT

In this section, we use our understanding of the Mayer-Vietoris spectral sequence in local cohomology to obtain results on the vanishing and structure of the local cohomology of a polynomial ring R over a field with support in an ideal I defining a linear subspace arrangement. We also establish a relation between these local cohomology modules and the connectedness dimension of the spectrum of R/I.

5.1. **The vanishing of the local cohomology.**

Theorem 5.1. *Under Setup 3.3, fix $i \in \mathbb{N}$. Then $H^i_I(R) = 0$ if and only if*

$$H_j(\Lambda_{i+j-h}(S), \Lambda_{i+j-h-1}(S); \Bbbk) = 0$$

for all $0 \le j \le n-i$, or equivalently, for all $j \in \mathbb{N}$.

Proof. By Corollary 4.10, $H^i_I(R) = 0$ if and only if for all $0 \le j \le n-i$, $E_2^{-j,i+j} = 0$, which happens if and only if $H_j(\Lambda_{i+j-h}(S), \Lambda_{i+j-h-1}(S); \Bbbk) = 0$ by Corollary 4.7(c). □

Theorem 5.1 yields the following concrete characterization of the cohomological dimension of a polynomial ring with support in an ideal defining a central linear subspace arrangement.

Corollary 5.2. *Under Setup 3.3,*

$$\operatorname{cd}(R, I) = n - \min\{i + j \mid i, j \in \mathbb{Z} \text{ and } \widetilde{H}_j(\Lambda_{d-i}(S); \Bbbk) \neq 0\}.$$

Proof. If $k = n - i$, Theorem 5.1 says that $H^{n-k}_I(R) = 0$ if and only if

$$H_j(\Lambda_{d-(k-j)}(S), \Lambda_{d-(k-j)-1}(S); \Bbbk) = 0$$

for all $0 \le j \le k$, or equivalently, for all $j \in \mathbb{Z}$. In other words, $H^{n-k}_I(R) \neq 0$ if and only if for some $i, j \in \mathbb{Z}$ such that $i + j = k$, $H_j(\Lambda_{d-i}(S), \Lambda_{d-i-1}(S); \Bbbk) \neq 0$. Hence

$$\begin{aligned}
\operatorname{cd}(R, I) &= \max\{n - (i + j) \mid i, j \in \mathbb{Z} \text{ and } H_j(\Lambda_{d-i}(S), \Lambda_{d-i-1}(S); \Bbbk) \neq 0\} \\
&= n - \min\{i + j \mid i, j \in \mathbb{Z} \text{ and } H_j(\Lambda_{d-i}(S), \Lambda_{d-i-1}(S); \Bbbk) \neq 0\} \\
&= n - \min\{i + j \mid i, j \in \mathbb{Z} \text{ and } \widetilde{H}_j(\Lambda_{d-i}(S); \Bbbk) \neq 0\}
\end{aligned}$$

where the last equality follows from Proposition 2.6. □

For $\Bbbk$ a field, we use $h_\bullet(-; \Bbbk)$ to denote $\dim_\Bbbk H_\bullet(-; \Bbbk)$, and analogously, $\widetilde{h}_\bullet(-; \Bbbk) = \dim_\Bbbk \widetilde{H}_\bullet(-; \Bbbk)$.

Example 5.3. We turn again to our running example, Example 2.2. One can check that the nonzero $h_k(\Lambda_t(S), \Lambda_{t-1}(S); \Bbbk) = \dim_\Bbbk H_k(\Lambda_t(S), \Lambda_{t-1}(S); \Bbbk)$ are the following.

$$\begin{aligned}
h_0(\Lambda_0(S), \Lambda_{-1}(S); \Bbbk) &= 4 \qquad & h_1(\Lambda_1(S), \Lambda_0(S); \Bbbk) &= 2 \\
h_1(\Lambda_2(S), \Lambda_1(S); \Bbbk) &= 3 \qquad & h_2(\Lambda_3(S), \Lambda_2(S); \Bbbk) &= 2
\end{aligned}$$

Applying Theorem 5.1, the first row of values tells us that $H^2_I(R)$ is not the zero module, while the second row says the same about $H^3_I(R)$. Moreover, these are the only nonzero local cohomology modules of R with support in I.

What's more, in the notation of Example 4.8, Corollary 4.10 guarantees the existence of short exact sequences of the following forms.

$$0 \to N_{12} \oplus N_{34} \oplus N_{56} \oplus N_{16} \to H^2_I(R) \to N_{126} \oplus N_{156} \to 0$$
$$0 \to N_{1234} \oplus N_{3456} \oplus N_{1346} \to H^3_I(R) \to N_{12346} \oplus N_{13456} \to 0$$

Theorem 5.1 also yields a vanishing condition for the simplicial homology of the codimension complexes.

Corollary 5.4. *Under Setup 3.3, for all integers $j > t \geq -1$,*

$$\widetilde{H}_j(\Lambda_t(S);\Bbbk) = H_j(\Lambda_t(S), \Lambda_{t-1}(S);\Bbbk) = 0.$$

Proof. First suppose that for some $j > t \geq -1$, $H_j(\Lambda_t(S), \Lambda_{t-1}(S);\Bbbk)$ is nonzero. Then by Theorem 5.1, $H^i_I(R) \neq 0$, where $i = t - j + h < h$, a contradiction. Hence

$$H_j(\Lambda_t(S), \Lambda_{t-1}(S);\Bbbk) = 0 \text{ for all } j > t \geq -1. \tag{5.1}$$

Now we induce on $t \geq -1$ to show that for all $j > t$, $\widetilde{H}_j(\Lambda_t(S);\Bbbk) = 0$. As $\Lambda_{-1}(S) = \{\varnothing\}$, $\widetilde{H}_j(\Lambda_{-1}(S);\Bbbk) = 0$ for $j \neq -1$, and the statement holds when $t = -1$. Now fix $t \geq 0$, and assume that $\widetilde{H}_j(\Lambda_{t-1}(S);\Bbbk) = 0$ for all $j \geq t$. By (5.1) and the long exact sequence in reduced relative simplicial homology with respect to $\Lambda_{t-1}(S) \subseteq \Lambda_t(S)$ (2.1), $\widetilde{H}_j(\Lambda_t(S);\Bbbk) \cong \widetilde{H}_j(\Lambda_{t-1}(S);\Bbbk)$ for all $j > t$. The inductive hypothesis completes the proof. □

5.2. **The dimension of the support of the local cohomology.**

Lemma 5.5. *Adopt Setup 3.3, and consider the Mayer-Vietoris spectral sequence in local cohomology with respect to $I_1, \ldots, I_\ell$. If p, q are integers for which $E_2^{-p,q} \neq 0$, then* $\dim \operatorname{Supp} E_2^{-p,q} = n - q$.

Proof. First suppose that J is an ideal of R that is generated by linear forms. Then $H^i_J(R)$ is nonzero only for $i = \operatorname{height}_R J$ (see Remark 3.1), in which case we claim that $\operatorname{Supp} H^i_J(R) = \mathbb{V}(J)$. To see this, first observe that $(H^i_J(R))_J \cong H^i_{JR_J}(R_J)$ is not zero since JR_J is the maximal ideal of the local ring R_J and $\dim R_J = \operatorname{height}_R J$. Moreover, given $P \in \operatorname{Spec}(R)$ such that $J \not\subseteq P$, there exists $x \in J \setminus P$, and its image in JR_P is a unit, so $(H^i_J(R))_P \cong H^i_{JR_P}(R_P) = 0$.

Observe that by Corollary 4.7(a), if $E_2^{-p,q}$ is not zero, then it is the direct sum over the $J \in \mathbb{J}(\Lambda_{q-h}(S) \setminus \Lambda_{q-h-1}(S))$ for which $H_p^J(\Lambda_{q-h}(S), \Lambda_{q-h-1}(S); H^q_J(R))$ is not zero, of which there is at least one. Hence the support of $E_2^{-p,q}$ is the union, over these J, of $\mathbb{V}(J)$. Moreover, each such J has height q by Remarks 2.3 and 3.1, and so $\dim \mathbb{V}(J) = n - q$. □

Theorem 5.6. *Under Setup 3.3, if $H^i_I(R) \neq 0$ for some $i \in \mathbb{N}$, then*

$$\dim \operatorname{Supp} H^i_I(R) = n - i - \rho,$$

where ρ is the least integer $0 \leq j \leq n - i$ such that $H_j(\Lambda_{i+j-h}(S), \Lambda_{i+j-h-1}(S);\Bbbk) \neq 0$.

Proof. First observe that by Theorem 5.1, ρ exists since $H^i_I(R) \neq 0$.

Consider the filtration $\{\mathcal{M}^i_k\}_{k=-1}^{n-i}$ of $H^i_I(R)$ ensured by Corollary 4.10, and for $k \geq 0$, let

$$\Theta_k = \{0 \leq j \leq k \mid H_j(\Lambda_{i+j-h}(S), \Lambda_{i+j-h-1}(S);\Bbbk) \neq 0\}.$$

We will induce on $0 \leq k \leq n - i$ to prove that $\mathcal{M}^i_k \neq 0$ if and only if $\Theta_k \neq \varnothing$, in which case

$$\dim \operatorname{Supp} \mathcal{M}^i_k = n - i - \min(\Theta_k).$$

Assuming this, the claim follows by taking $k = n - i$, since $H^i_I(R) = \mathcal{M}^i_{n-i}$.

For the case that $k = 0$, first observe that by Corollary 4.7(c), $\mathcal{M}^i_0 \neq 0$ if and only if $H_0(\Lambda_{i-h}(S), \Lambda_{i-h-1}(S);\Bbbk) \neq 0$. This holds if and only if $\Theta_0 \neq \varnothing$, which in fact happens if and only if $\min(\Theta_k)$ exists and equals 0. Moreover, if $\mathcal{M}^i_0 \neq 0$, since $\mathcal{M}^i_0 = E_2^{0,i}$, the dimension of its support is $n - i$ by Lemma 5.5, which agrees with $n - i - \Theta_0$ since $\min(\Theta_0) = 0$.

Now, fix $1 \le k \le n-i$, and assume that $\mathcal{M}^i_{k-1} \neq 0$ if and only if $\Theta_{k-1} \neq \varnothing$, in which case $\dim \operatorname{Supp} \mathcal{M}^i_{k-1} = n - i - \min(\Theta_{k-1})$. The short exact sequence

$$0 \to \mathcal{M}^i_{k-1} \to \mathcal{M}^i_k \to E_2^{-k,i+k} \to 0$$

guarantees that

$$\operatorname{Supp} \mathcal{M}^i_k = \operatorname{Supp} \mathcal{M}^i_{k-1} \cup \operatorname{Supp} E_2^{-k,i+k}.$$

First suppose that $\mathcal{M}^i_{k-1} = 0$, so that $\mathcal{M}^i_k \cong E_2^{-k,i+k}$, which is not zero if and only if $H_j(\Lambda_{i+k-h}(S), \Lambda_{i+k-h-1}(S); \Bbbk) \neq 0$ by Corollary 4.7(c). By the inductive hypothesis, $H_j(\Lambda_{i+j-h}(S), \Lambda_{i+j-h-1}(S); \Bbbk) = 0$ for all $0 \le j \le k-1$, so $\Theta_k \neq \varnothing$ if and only if $\mathcal{M}^i_k \neq 0$, in which case $\min(\Theta_k) = k$. By Lemma 5.5, in this case, $\dim \operatorname{Supp} \mathcal{M}^i_k = n - (i+k)$, which equals $n - i - \min(\Theta_k)$, and the inductive step holds.

Now assume that $\mathcal{M}^i_{k-1} \neq 0$, so that by the inductive hypothesis, $\Theta_{k-1} \neq \varnothing$ and $H_j(\Lambda_{i+j-h}(S), \Lambda_{i+j-h-1}(S); \Bbbk) \neq 0$ when $j = \min(\Theta_{k-1})$. Since $0 \le \min(\Theta_{k-1}) \le k-1$, $\Theta_k \neq \varnothing$ as well, and $\min(\Theta_k)$ is at most $k-1$, so $\min(\Theta_k) = \min(\Theta_{k-1})$. If $E_2^{-k,i+k} = 0$, then $\mathcal{M}^i_k = \mathcal{M}^i_{k-1}$, so by the inductive hypothesis, $\dim \operatorname{Supp} \mathcal{M}^i_k = n - i - \min(\Theta_{k-1}) = n - i - \min(\Theta_k)$, and we are done. Otherwise, $\dim \operatorname{Supp} \mathcal{M}^i_k$ is the maximum of the dimensions of the supports of $\mathcal{M}^i_{k-1}$ and of $E_2^{-k,i+k}$, but since by Lemma 5.5,

$$\dim \operatorname{Supp} E_2^{-k,i+k} = n - (i+k) < n - i - \min(\Theta_{k-1}) = \dim \operatorname{Supp} \mathcal{M}^i_{k-1},$$

$\dim \operatorname{Supp} \mathcal{M}^i_k = n - i - \min(\Theta_{k-1}) = n - i - \min(\Theta_k)$ in this case as well. □

Corollary 5.7. *Under Setup 3.3, for any $i \in \mathbb{Z}$, the injective dimension of $H^i_I(R)$ is at most*

$$n - i - \min\{0 \le j \le n-i \mid H_j(\Lambda_{i+j-h}(S), \Lambda_{i+j-h-1}(S); \Bbbk) \neq 0\}.$$

Proof. This follows from Theorem 5.6 since $H^i_I(R) \cong H^i_{IR_\mathfrak{m}}(R_\mathfrak{m})$, and the injective dimension of the local cohomology of a regular local ring of equal characteristic is bounded above by the dimension of its support [HS93, Corollary 3.9],[Lyu93, Corollary 3.6]. □

5.3. **An application to connectedness dimension.** The goal of this section is to relate the local cohomology with support in an ideal defining a linear subspace arrangement to connectedness dimension. The statement of Theorem 5.11 does not reference the codimension complexes nor simplicial homology.

Before our first result, which characterizes the connectedness dimension in terms of the codimension complexes, we recall a couple relevant definitions.

Definition 5.8. The *connectedness dimension* $\mathrm{c}(A)$ of a Noetherian ring A is the minimum value of $\dim(X)$ among all closed subsets $X \subsetneq \operatorname{Spec}(A)$ for which $\operatorname{Spec}(A) \setminus X$ is disconnected in the Zariski topology.

Definition 5.9. The *subdimension* of a Noetherian ring A, denoted $\operatorname{sdim}(A)$, is the minimum value of $\dim(A/P)$ as P varies over its minimal primes.

Theorem 5.10 (Cf. [NSW19, Proposition 2.5])**.** *Let A be a Noetherian ring of dimension $d \ge 2$, and fix $0 \le t \le d-1$. Then $\mathrm{c}(A) \ge t$ if and only if $t \le \operatorname{sdim}(A)$ and $\Lambda_{d-t}(A)$ is connected. Equivalently,*

$$\mathrm{c}(A) = \max\{0 \le t \le \operatorname{sdim}(A) \mid \widetilde{H}_0(\Lambda_{d-t}(A); \Bbbk) = 0\}.$$

Proof. First notice that the case when $t = 0$ always holds, since $\mathrm{c}(A) \geq 0$, $\operatorname{sdim}(A) \geq 0$, and $\Lambda_d(A)$ is a full simplicial complex.

Now, let $1 \leq t \leq d-1$. By [Har62, Proposition 1.1], $\operatorname{Spec}(A) \setminus X$ is connected for all $X \subseteq \operatorname{Spec}(A)$ closed of dimension less than t if and only if each pair of minimal primes P, Q of A admits a sequence of minimal primes

$$P = P_1, P_2, \ldots, P_r = Q$$

such that $\dim\left(A/(P_j + P_{j+1})\right) \geq t$ for each $1 \leq j < r$. We will complete the proof by showing that this is equivalent to the following pair of conditions.

(1) The vertices of $\Lambda_{d-t}(A)$ correspond to all minimal primes of A, i.e., $t \leq \operatorname{sdim}(A)$.
(2) The complex $\Lambda_{d-t}(A)$ is connected.

Indeed, if for each pair of minimal primes P, Q of A, such a sequence exists, then there is a sequence from each P to itself. If P does not correspond to a vertex of $\Lambda_{d-t}(A)$ then $\dim(A/P) = \dim(A/(P+P)) < t$, while for any minimal prime $Q \neq P$, $\dim\left(A/(P+Q)\right) < \dim(A/P) < t$ as well, a contradiction. Hence the first condition holds.

Now, $\Lambda_{d-t}(A)$ is connected if and only if there is a path between any two of its vertices. A pair of its vertices correspond to an arbitrary pair of minimal primes P, Q, and a path between the vertices in (the 1-skeleton) of $\Lambda_{d-t}(A)$ is equivalent to a sequence $P = P_1, P_2, \ldots, P_r = Q$ of minimal primes such that $\dim\left(A/(P_j + P_{j+1})\right) \geq t$ for each $1 \leq j < r$. This establishes the reverse implication as well. □

Notice that in the statement below, the condition on the dimension of the support of $H_I^{n-i}(R)$ when $i = 1$ forces $H_I^{n-1}(R)$ to be the zero module.

Theorem 5.11. *Under Setup 3.3, assume that $d \geq 2$, and fix $1 \leq j \leq \operatorname{sdim}(S)$. If*

$$\dim \operatorname{Supp} H_I^{n-i}(R) \leq i - 2$$

for each $1 \leq i \leq j$, then $\mathrm{c}(S) \geq j$.

Proof. By Theorem 5.6, for every integer i, $\dim \operatorname{Supp} H_I^{n-i}(R) = i - \eta_i$, where

$$\eta_i = \min\{0 \leq r \leq i \mid H_r(\Lambda_{d-i+r}(S), \Lambda_{d-i+r-1}(S); \Bbbk) \neq 0\}.$$

Given $1 \leq i \leq j$, since

$$i - \eta_i = \dim \operatorname{Supp} H_I^{n-i}(R) \leq i - 2,$$

this means that $\eta_i \geq 2$. Hence for $1 \leq i \leq j$,

$$H_0(\Lambda_{d-i}(S), \Lambda_{d-i-1}(S); \Bbbk) = H_1(\Lambda_{d-i+1}(S), \Lambda_{d-i}(S); \Bbbk) = 0.$$

Re-indexing the subscripts of the codimension complexes appearing in the first homology, we have that $H_0(\Lambda_{d-i+1}(S), \Lambda_{d-i}(S); \Bbbk) = 0$ for all $2 \leq i \leq j+1$, while this homology also vanishes for $i = 1$ since $H_0(\Lambda_d(S), \Lambda_{d-1}(S); \Bbbk) \cong \widetilde{H}_{-1}(\Lambda_{d-1}(S); \Bbbk)$ by Remark 2.4, and this is nonzero since $d \geq 2$. We conclude that for $r = 0, 1$ and $1 \leq i \leq j$,

$$H_r(\Lambda_{d-i+1}(S), \Lambda_{d-i}(S); \Bbbk) = 0. \tag{5.2}$$

Our next goal is to induce on i to prove that for all $1 \leq i \leq j$, $\widetilde{H}_0(\Lambda_{d-i}(S); \Bbbk) = 0$. The vanishing condition (5.2) and long exact sequence in reduced relative homology (2.1) with respect to $\Lambda_{d-i}(S) \subseteq \Lambda_{d-i+1}(S)$ imply that for $1 \leq i \leq j$,

$$\widetilde{H}_0(\Lambda_{d-i}(S); \Bbbk) \cong \widetilde{H}_0(\Lambda_{d-i+1}(S); \Bbbk).$$

When $i = 1$, $\Lambda_{d-i+1}(S) = \Lambda_d(S)$ is a full simplicial complex, forcing both to vanish. Similarly, assuming that $\widetilde{H}_0(\Lambda_{d-i}(S);\Bbbk) = 0$ for some $1 \le i < j$, the long exact sequence with respect to $\Lambda_{d-i-1}(S) \subseteq \Lambda_{d-i}(S)$ ensures that $\widetilde{H}_0(\Lambda_{d-(i+1)}(S);\Bbbk)$ also vanishes. Hence $\widetilde{H}_0(\Lambda_{d-i}(S);\Bbbk) = 0$ for every $1 \le i \le j$. Moreover, $j \le \operatorname{sdim}(A)$, so an application of Theorem 5.10 completes the proof. □

5.4. **The length of the local cohomology as a module over the Weyl algebra.** Suppose that $R = \Bbbk[x_1, \dots, x_n]$, where $\Bbbk$ is a field of characteristic zero. Recall that the Weyl algebra $\mathcal{D}(R, \Bbbk)$ is the noncommutative ring $\Bbbk\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n\rangle$ modulo the two-sided ideal generated by, for $1 \le i, j \le n$, $x_i x_j - x_j x_i$, $\partial_i\partial_j - \partial_j\partial_i$, and $\partial_i x_i - x_i\partial_i - 1$, and for $i \ne j$, $\partial_i x_j - x_j\partial_i$.

Then R is naturally a $\mathcal{D}(R, \Bbbk)$-module. Given $g \in R$, for $1 \le i \le n$, $x_i \bullet g = x_i g$ and $\partial_i \bullet g = \frac{\partial g}{\partial x_i}$. Moreover, for any $f \in R$, the localization R_f is also a $\mathcal{D}(R, \Bbbk)$-module via, for $k \ge 0$, $x_i \bullet \frac{g}{f^k} = \frac{x_i g}{f^k}$ and $\partial_i \bullet \frac{g}{f^k} = \frac{1}{f^k}\frac{\partial g}{\partial x_i} - \frac{kg}{f^{k+1}}\frac{\partial f}{\partial x_i}$.

After choosing generators for an ideal I of R, the modules of the associated Čech-like complex defining the local cohomology of R with support in I are each direct sums of modules of the form R_f which are $\mathcal{D}(R, \Bbbk)$-modules. It is not difficult to see that the differentials are $\mathcal{D}(R, \Bbbk)$-linear, making it a complex of $\mathcal{D}(R, \Bbbk)$-modules. Hence the cohomology modules $H^i_I(R)$ are again $\mathcal{D}(R, \Bbbk)$-modules, and have finite length as $\mathcal{D}(R, \Bbbk)$-modules by [Lyu93].

Lemma 5.12. *Suppose $R = \Bbbk[x_1, \dots, x_n]$, where $\Bbbk$ is a field of characteristic zero, and consider an ideal J of R that is generated by linear forms. If h is the height of J, then $H^h_J(R)$ is a simple $\mathcal{D}(R, \Bbbk)$-module.*

Proof. We can reduce to the case where J is generated by variables by Remark 3.1, and without loss of generality, assume that $J = \langle x_1, \dots, x_h\rangle$. For $\mathbf{v} \in \mathbb{N}^h_{>0}$, let $\mathbf{x}^{\mathbf{v}}$ denote $x_1^{v_1} \cdots x_h^{v_h}$, and given $\mathbf{u} \in \mathbb{N}^{n-h}$, let $\mathbf{y}^{\mathbf{u}} = x_{h+1}^{u_1} \cdots x_n^{u_{n-h}}$. By Remark 4.6, the elements $\frac{\mathbf{y}^{\mathbf{u}}}{\mathbf{x}^{\mathbf{v}}}$ form a $\Bbbk$-vector space basis for $H^h_J(R)$. Hence, fixing a nonzero submodule N of $H^h_J(R)$, it suffices to show that N contains each of these elements.

Fix a nonzero element $z \in N$, so that $z = \sum_{\mathbf{u}\in U, \mathbf{v}\in V} \frac{c_{\mathbf{u},\mathbf{v}}\mathbf{y}^{\mathbf{u}}}{\mathbf{x}^{\mathbf{v}}}$ for some finite sets $V \subseteq \mathbb{N}^h_{>0}$ and $U \subseteq \mathbb{N}^{n-h}$, and elements $c_{\mathbf{u},\mathbf{v}} \in \Bbbk$. Moreover, fix $\mathbf{u}' \in U$ with maximal coordinate sum among the elements in U, and let $\rho = \prod_{i=1}^{n-h} \partial_{h+i}^{u'_i} \in \mathcal{D}(R, \Bbbk)$. Then for each $\mathbf{u} \in U$ different from $\mathbf{u}'$, $u_i < u'_i$ for some $1 \le i \le n-h$, which implies that $\rho \bullet \frac{\mathbf{y}^{\mathbf{u}}}{\mathbf{x}^{\mathbf{v}}} = 0$ since $\partial_{h+i}^{u'_i}$ appears in ρ, and $x_{h+i}^{u_i}$ is a term of the product $\mathbf{y}^{\mathbf{u}}$. On the other hand, $a := \rho \bullet \mathbf{y}^{\mathbf{u}'}$ is in $\Bbbk$. Hence

$$\rho \bullet z = \left(\prod_{i=1}^{n-h} \partial_{h+i}^{u'_i}\right) \bullet \left(\sum_{\mathbf{u}\in U, \mathbf{v}\in V} \frac{c_{\mathbf{u},\mathbf{v}}\mathbf{y}^{\mathbf{u}}}{\mathbf{x}^{\mathbf{v}}}\right) = \sum_{\mathbf{v}\in V'} \frac{a c_{\mathbf{u}',\mathbf{v}}}{\mathbf{x}^{\mathbf{v}}} \in N,$$

where $V' = \{\mathbf{v} \in V \mid c_{\mathbf{u}',\mathbf{v}} \ne 0\}$, so that $z' = \sum_{\mathbf{v}\in V'} \frac{c_{\mathbf{u}',\mathbf{v}}}{\mathbf{x}^{\mathbf{v}}}$ must also be in N.

Now, let $\mathbf{v}'$ be an element of maximal norm among all elements of V', and let $\sigma = \mathbf{x}^{\mathbf{v}'_i - 1} \in R$. For $\mathbf{v} \in V'$ such that $\mathbf{v} \ne \mathbf{v}'$, there exists $1 \le i \le h$ for which $v_i \le v'_i - 1$, so that $\sigma \bullet \frac{1}{\mathbf{x}^{\mathbf{v}}} = 0$, while $\sigma \bullet \frac{1}{\mathbf{x}^{\mathbf{v}'}} = \frac{1}{x_1 x_2 \cdots x_h}$. Hence $\sigma \bullet z' = \frac{c_{\mathbf{u}',\mathbf{v}'}}{x_1 \cdots x_h} \in N$, and so $\frac{1}{x_1 \cdots x_h} \in N$ as well.

Finally, given arbitrary $\mathbf{v} \in \mathbb{N}^h_{>0}$ and $\mathbf{u} \in \mathbb{N}^{n-h}$, the operator $\mathbf{y}^{\mathbf{u}} \prod_{i=1}^h \partial_i^{v_i - 1} \in \mathcal{D}(R, \Bbbk)$ applied to $\frac{1}{x_1 \cdots x_h} \in N$ yields a $\Bbbk$-multiple of $\frac{\mathbf{y}^{\mathbf{u}}}{\mathbf{x}^{\mathbf{v}}}$, so $\frac{\mathbf{y}^{\mathbf{u}}}{\mathbf{x}^{\mathbf{v}}} \in N$, and $N = H^h_J(R)$. $\square$

Remark 5.13. Setup 3.3, if $\Bbbk$ has characteristic zero, then by [ÀGZ03, Corollary 1.3], the filtration $\{\mathcal{M}^i_j\}_{j=-1}^{n-i}$ of $H^i_I(R)$ by R-modules ensured by Corollary 4.10 is a filtration of $\mathcal{D}(R, \Bbbk)$-modules.

Theorem 5.14. *Adopt Setup 3.3, and suppose that $\Bbbk$ has characteristic zero. For each integer i, the length of $H^i_I(R)$ as a $\mathcal{D}(R, \Bbbk)$-module is*

$$\sum_{j=0}^{n-i} h_j(\Lambda_{i+j-h}(S), \Lambda_{i+j-h-1}(S); \Bbbk).$$

Proof. Given a $\mathcal{D}(R, \Bbbk)$-module N, let $\ell(N)$ denote its length as a $\mathcal{D}(R, \Bbbk)$-module. If $\{\mathcal{M}^i_j\}_{j=-1}^{n-i}$ is the filtration of $H^i_I(R)$ ensured by Corollary 4.10, then it is also a filtration of $\mathcal{D}(R, \Bbbk)$-modules by Remark 5.13, so $\ell(E_2^{-j,i+j}) = \ell(\mathcal{M}^i_j) - \ell(\mathcal{M}^i_{j-1})$ for each $0 \le j \le n - i$. Therefore,

$$\sum_{j=0}^{n-i} \ell(E_2^{-j,i+j}) = \sum_{j=0}^{n-i} \left(\ell(\mathcal{M}^i_j) - \ell(\mathcal{M}^i_{j-1}) \right) = \ell(H^i_I(R))$$

since $\mathcal{M}^i_{-1} = 0$ and $\mathcal{M}^i_{n-i} = H^i_I(R)$.

Thus, it suffices to show that $\ell(E_2^{-j,i+j}) = h_j(\Lambda_{i+j-h}(S), \Lambda_{i+j-h-1}(S); \Bbbk)$. Toward this, if $\Omega = \Lambda_{i+j-h}(S) \setminus \Lambda_{i+j-h-1}(S)$, then

$$\ell(E_2^{-j,i+j}) = \sum_{J \in \mathbb{J}(\Omega)} \ell(H_j(C^J_\bullet(\Lambda_{i+j-h}(S), \Lambda_{i+j-h-1}(S); H^{i+j}_J(R))))$$

by Corollary 4.7. Setting $a_J = h_j(C^J_\bullet(\Lambda_{i+j-h}(S), \Lambda_{i+j-h-1}(S); \Bbbk))$, Remark 2.3 ensures that $H^{i+j}_J(R) \neq 0$ and Remark 4.6 ensures that $H_j(C^J_\bullet(\Lambda_{i+j-h}(S), \Lambda_{i+j-h-1}(S); H^{i+j}_J(R))) \cong \left(H^{i+j}_J(R)\right)^{\oplus a_J}$, which has $\mathcal{D}(R, \Bbbk)$-module length a_J since the local cohomology module appearing is a simple $\mathcal{D}(R, \Bbbk)$-module by Lemma 5.12. Hence, $\ell(E_2^{-j,i+j}) = \sum_{J \in \mathbb{J}(\Omega)} a_J$, which equals $h_j(\Lambda_{i+j-h}(S), \Lambda_{i+j-h-1}(S); \Bbbk)$ by Theorem 3.8, completing the proof. $\square$

Example 5.15. We return again to Example 2.2, recalling that, we found in Example 5.3 that $H^i_I(R) \neq 0$ only for $i = 2$ and $i = 3$. Using the nonzero values of $h_k(\Lambda_t(S), \Lambda_{t-1}(S); \Bbbk)$ recorded there, we find by Theorem 5.14 that if $\Bbbk$ has characteristic zero, then the $\mathcal{D}(R, \Bbbk)$-module length of $H^2_I(R)$ is $h_0(\Lambda_0(S), \Lambda_{-1}(S); \Bbbk) + h_1(\Lambda_1(S), \Lambda_0(S); \Bbbk) = 4 + 2 = 6$, and that of $H^3_I(R)$ equals $h_1(\Lambda_2(S), \Lambda_1(S); \Bbbk) + h_2(\Lambda_3(S), \Lambda_2(S); \Bbbk) = 3 + 2 = 5$.

5.5. Applications to general linear subspace arrangements.

Definition 5.16 (General central linear subspace arrangement)**.** Suppose that an ideal I in an n-dimensional polynomial ring over a field defines a central linear subspace arrangement. If $P_1, \dots, P_\ell$ are the minimal primes of I, we say that I defines a *general (central) linear subspace arrangement* if for every subset $A \subseteq [\ell]$,

$$\text{height}\left(\sum_{i \in A} P_i\right) = \min\left\{n, \sum_{i \in A} \text{height } P_i\right\}.$$

Remark 5.17. Under Setup 3.3, suppose that I defines a general linear subspace arrangement. Then when $q \neq n$, all maps $\phi_1^{-p,q}(u)$ on the E_1-page of the Mayer-Vietoris spectral sequence with respect to $I_1, \dots, I_\ell$ must be the zero map. Indeed, if $1 \le i_1, \dots, i_{j+1} \le \ell$ are distinct integers, then either $\operatorname{height}_R(I_{i_1} + \cdots + I_{i_{j+1}}) > \operatorname{height}_R(I_{i_1} + \cdots + I_{i_j})$, or $I_{i_1} + \cdots + I_{i_j} = I_{i_1} + \cdots + I_{i_{j+1}} = \mathfrak{m}$.

Hence $E_2^{-p,q} = E_1^{-p,q}$ whenever $q \neq n$, and since $E_1^{-p,q} = 0$ for $q > n$, and by Proposition 4.3, and Corollary 4.7(b), $h_p(\Lambda_{q-h}(S), \Lambda_{q-h-1}(S); \Bbbk) = \chi_{p,q}(S)$, where $\chi_{p,q}(S)$ is the number of p-faces of $\Lambda_{q-h}(S)$, or in other words, the number of sums of exactly $p+1$ minimal primes of I whose height is q; in particular, $\chi_{p,q}(S)$ does not depend on the characteristic of $\Bbbk$.

Hence for general linear subspace arrangements, $\chi_{j,i+j}(S) = h_j(\Lambda_{i+j-h}(S), \Lambda_{i+j-h-1}(S); \Bbbk)$ for each $0 \le j < n-i$. As such, the sum presented in Theorem 5.14 can be rewritten as

$$h_{n-i-1}(\Lambda_{d-1}(S); \Bbbk) + \sum_{j=0}^{n-i-1} \chi_{j,i+j}(S)$$

since $h_{n-i}(\Lambda_d(S), \Lambda_{d-1}(S); \Bbbk) = h_{n-i-1}(\Lambda_{d-1}(S); \Bbbk)$ by Remark 2.4. Similarly, in Theorems 5.1 and 5.6, when determining the vanishing of $H_j(\Lambda_{i+j-h}(S), \Lambda_{i+j-h-1}(S); \Bbbk)$, one can instead check the vanishing of $\chi_{j,i+j}(S)$ for $0 \le j < n-i$, or of $h_{n-i-1}(\Lambda_{d-1}(S); \Bbbk)$ when $j = n-i$. Theorem 6.10 on Lyubeznik numbers appearing in the next section can be analogously simplified for general arrangements.

Theorem 5.18. *Under Setup 3.3, assume that that I defines a general linear subspace arrangement, $h > 1$, and S is equidimensional. Let $r = \lceil n/h \rceil$. Then for $1 \le j \le r-1$,*

$$H_I^{jh-j+1}(R) \cong \bigoplus_{1 \le i_1 < \cdots < i_j \le \ell} H^{jh}_{I_{i_1} + \cdots + I_{i_j}}(R),$$

and

$$H_I^{n-r+1}(R) \cong E_R(\Bbbk)^{\oplus \binom{\ell-1}{r-1}}.$$

Moreover, all other $H_I^i(R) = 0$.

Proof. First suppose that $jh < n$ for some integer $j \ge 1$, i.e., $j < r$. Then the only nonzero term in the jh-th row of the E_1-page of the Mayer-Vietoris spectral sequence with respect to $I_1, \dots, I_\ell$ is $E_1^{-j+1,jh}$, which is the direct sum of $H^{jh}_{I_{i_1} + \cdots + I_{i_j}}(R)$, ranging over all $1 \le i_1 < \cdots < i_j \le \ell$. Since I defines a general linear subspace arrangement, for $1 \le j < r$,

$$E_2^{-j+1,jh} = \bigoplus_{1 \le i_1 < \cdots < i_j \le \ell} H^{jh}_{I_{i_1} + \cdots + I_{i_j}}(R) \tag{5.3}$$

by Remark 5.17.

Now consider the n-th row of the E_1-page. We have that $E_1^{-j+1,n} \neq 0$ if and only if $j \le \ell$ and for any $1 \le i_1 < \cdots < i_j \le \ell$, $\sum_{k=1}^{j} \operatorname{height}_R(I_{i_k}) = n$. Since I defines a general arrangement, this is equivalent to the condition that $\operatorname{height}_R(I_{i_1}) + \cdots + \operatorname{height}_R(I_{i_j}) \ge n$, and since S is equidimensional, this happens if and only if $jh \ge n$, i.e., when $j \ge r = \lceil n/h \rceil$. Moreover, for $r \le j \le \ell$, $E_1^{-j+1,n} \cong (H^n_{\mathfrak{m}}(R))^{\binom{\ell}{j}} \cong (E_R(\Bbbk))^{\oplus \binom{\ell}{j}}$, since there are $\binom{\ell}{j}$ choices of integers $1 \le i_1 < \cdots < i_j \le \ell$.

Notice that by Theorem 4.4, $E_1^{-j+1,n} \cong C_{j-1}(\Lambda_d(S); E_R(\Bbbk))$ for $r \le j \le \ell$. since $C_{j-1}(\Lambda_{d-1}(S); E_R(\Bbbk)) = 0$, by virtue of the fact that every sum of j minimal primes of R has height n. Moreover, for $r+1 \le j \le \ell$, $\phi_1^{-j+1,n} : E_1^{-j+1,n} \to E_1^{-j+2,n}$ is simply $\partial_{j-1} \otimes_R E_R(\Bbbk)$, for $\partial_{j-1} : C_{j-1}(\Lambda_d(S); \Bbbk) \to C_{j-2}(\Lambda_d(S); \Bbbk)$, by Remark 4.5. Since $\Lambda_d(S)$ is the full simplicial complex on ℓ vertices, $E_2^{-j+1,n} = 0$ for all $r+1 \le j \le \ell$.

We claim that $E_2^{-r+1,n} \cong E_R(\Bbbk)^{\oplus \binom{\ell-1}{r-1}}$. Since $\ker(\phi_1^{-r+1,n}) = E_1^{-r+1,n} \cong (E_R(\Bbbk))^{\oplus \binom{\ell}{r}}$ as observed earlier, it suffices to show that $\dim_{\Bbbk} \operatorname{Im}(\partial_r) = \binom{\ell-1}{r}$, since $\binom{\ell}{r} - \binom{\ell-1}{r} = \binom{\ell-1}{r-1}$. Indeed, it follows by induction on $j \ge 0$ that $\dim_{\Bbbk} \operatorname{Im}(\partial_j) = \binom{\ell-1}{j}$, since $\dim_{\Bbbk} \operatorname{Im}(\partial_0) = 1 = \binom{\ell-1}{0}$, and assuming $\dim_{\Bbbk} \operatorname{Im}(\partial_j) = \binom{\ell-1}{j}$ for some $j \ge 0$, we have that $\dim_{\Bbbk} \ker(\partial_j) = \dim C_j(\Lambda_d(S); \Bbbk) - \dim_{\Bbbk} \operatorname{Im}(\partial_j) = \binom{\ell}{j} - \binom{\ell-1}{j} = \binom{\ell-1}{j-1}$.

Hence the nonzero terms on the E_2-page are $E_2^{-j+1,jh} = H^{jh}_{I_{i_1}+\cdots+I_{i_j}}(R)$ for $1 \le j < r$, and $E_2^{-r+1,n} \cong (E_R(\Bbbk))^{\oplus \binom{\ell-1}{r-1}}$. Moreover, no two lie on the same diagonal: Given integers $j < k$, $jh - j + 1 = j(h-1) + 1 = k(h-1) + 1 = kh - k + 1$, since $h > 1$. Moreover, if $j < r$, then

$$jh - j + 1 = j(h-1) + 1 \le (r-1)(h-1) + 1 = rh - r - h + 1 < (n+h) - r - h + 1 = n - r + 1$$

since $n/h \le r < n/h + 1$, so $n \le rh < n + h$.

Thus, $H_I^{n-r+1}(R) \cong E_2^{-r+1,n} \cong (E_R(\Bbbk))^{\oplus \binom{\ell-1}{r-1}}$, and for $1 \le j < r$, $H_I^{jh-j+1}(R) \cong E_2^{-j+1,jh}$, and the second claim follows by (5.3) and all other $H_I^i(R) = 0$. □

6. Lyubeznik Numbers of a Central Linear Subspace Arrangement

Suppose that $(T, \mathfrak{n}, \Bbbk)$ is a local ring containing a field, and $T \cong A/J$ for some n-dimensional regular local ring $(A, \mathfrak{m}, \Bbbk)$ containing a field. Given $i, j \in \mathbb{Z}$, the corresponding *Lyubeznik number* of A is defined as

$$\lambda_{ij}(T) = \dim_{\Bbbk} \operatorname{Ext}^i_A(\Bbbk, H_J^{n-j}(A))$$

and is independent of the choice of A and J. In other words, $\lambda_{ij}(T)$ is the i-th Bass number of $H_J^{n-j}(A)$ with respect to $\mathfrak{m}$: If $E_A(\Bbbk)$ denotes the injective hull of $\Bbbk$ as an A-module, then $\lambda_{ij}(T)$ counts the copies of $E_A(\Bbbk)$ in the i-th term of a minimal injective resolution of $H_J^{n-j}(A)$. Each Lyubeznik number is finite. Moreover, $\lambda_{ij}(T) = 0$ whenever $i > j$, and if $d = \dim T$ then $\lambda_{dd}(T) \ne 0$, while $\lambda_{ij}(T) = 0$ if $i > d$ or $j > d$ [Lyu93, Theorem-Definition 4.1, Properties (4.4)].

Remark 6.1. Suppose that J is an ideal of an n-dimensional regular local ring $(A, \mathfrak{m}, \Bbbk)$ containing a field. By [Lyu06, Lemma 2.2], $H^i_{\mathfrak{m}}(H_J^{n-j}(A))$ is isomorphic to a direct sum of $\lambda_{ij}(A/J)$-many copies of $E_A(\Bbbk)$, the injective hull of $\Bbbk$ as an A-module.

Remark 6.2. Let J be an ideal of an n-dimensional regular local ring $(A, \mathfrak{m}, \Bbbk)$ containing a field, and suppose that $H^i_J(A) \ne 0$ if and only if $i = \operatorname{height}_A(J)$. Then the spectral sequence $E_2^{p,q} = H^p_{\mathfrak{m}}(H^q_J(A)) \underset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{m}}(A)$ [Har67, Proposition 1.4] has only one nonzero row, so degenerates on the E_2-page. Hence if $h = \operatorname{height}_A(J)$ and $d = \dim(A/J) = n - h$, then $H^d_{\mathfrak{m}}(H^h_J(A)) \cong H^n_{\mathfrak{m}}(A) \cong E_A(\Bbbk)$, and all other $H^p_{\mathfrak{m}}(H^q_J(A)) = 0$. Thus, by Remark 6.1, and $\lambda_{dd}(A/J) = 1$, where $d = \dim(A/J)$, and $\lambda_{ij}(A/J) = 0$ otherwise.

In the following statement, and the remainder of the paper, we use $h_\bullet(-;\Bbbk)$ to denote $\dim_\Bbbk H_\bullet(-;\Bbbk)$.

Proposition 6.3. *Adopt Setup 3.3, and let $\mathfrak{m}$ denote the homogeneous maximal ideal of R. Consider the Mayer-Vietoris spectral sequence in local cohomology with respect to $I_1, \dots, I_\ell$. Then for $p, q \in \mathbb{Z}$, $H^{n-q}_\mathfrak{m}(E_2^{-p,q}) \cong E_R(\Bbbk)^{\oplus a}$, where $a = h_p(\Lambda_{q-h}(S), \Lambda_{q-h-1}(S); \Bbbk)$, and $H^k_\mathfrak{m}(E_2^{-p,q}) = 0$ if $k \neq n-q$.*

Proof. By Corollary 4.7, for any $i \in \mathbb{Z}$,

$$H^i_\mathfrak{m}(E_2^{-p,q}) \cong \bigoplus_{J \in \mathbb{J}(\Omega)} H^i_\mathfrak{m}\left(H_p\left(C^J_\bullet(\Lambda_{q-h}(S), \Lambda_{q-h-1}(S); H^q_J(R)\right)\right),$$

where $\Omega = \Lambda_{q-h}(S) \setminus \Lambda_{q-h-1}(S)$. Remark 4.6 allows us to rewrite the module indexed by $J \in \mathbb{J}(\Omega)$ as $H^i_\mathfrak{m}(H^q_J(R)^{\oplus a_J}) \cong H^i_\mathfrak{m}(H^q_J(R))^{\oplus a_J}$, where $a_J = \dim_\Bbbk H_p(C^J_\bullet(\Lambda_{q-h}(S), \Lambda_{q-h-1}(S); \Bbbk)$. Since $\operatorname{height}_R(J) = q$ by definition of Ω (see Remark 2.3), we have that

$$H^i_\mathfrak{m}(H^q_J(R)) = H^i_{\mathfrak{m}R_\mathfrak{m}}(H^q_{JR_\mathfrak{m}}(R_\mathfrak{m})) \cong E_{R_\mathfrak{m}}(\Bbbk)^{\oplus c} = E_R(\Bbbk)^{\oplus c}$$

where $c = \lambda_{i,n-q}(R_\mathfrak{m}/JR_\mathfrak{m})$ by Remark 6.1. Now, by Remark 6.2, $c = 1$ if $i = d$ and $q = h$, and $c = 0$ otherwise.

In summary, $H^i_\mathfrak{m}(E_2^{-p,q}) = 0$ if $i \neq d$ or $q \neq h$, and $H^d_\mathfrak{m}(E_2^{-p,q}) \cong \bigoplus_{J \in \mathbb{J}(\Omega)} E_R(\Bbbk)^{\oplus a_J} \cong E_R(\Bbbk)^{\oplus a}$, where

$$a = \sum_{J \in \mathbb{J}(\Omega)} a_J = \sum_{J \in \mathbb{J}(\Omega)} h_p(C^J_\bullet(\Lambda_{q-h}(S), \Lambda_{q-h-1}(S); \Bbbk) = h_p(\Lambda_{q-h}(S), \Lambda_{q-h-1}(S); \Bbbk),$$

where the last equality follows from Theorem 3.8. □

In the following preliminary results, we often make use of the following exact sequences.

Remark 6.4. Under Setup 3.3, and given $i \in \mathbb{Z}$, let $\{\mathcal{M}^i_j\}$ be the filtration of $H^i_I(R)$ ensured by Corollary 4.10. Then for any $0 \leq j \leq n-i$, the short exact sequence

$$0 \to \mathcal{M}^i_{j-1} \to \mathcal{M}^i_j \to E_2^{-j,i+j} \to 0$$

induces a long exact sequence of the form

$$\cdots \to H^k_\mathfrak{m}(\mathcal{M}^i_{j-1}) \to H^k_\mathfrak{m}(\mathcal{M}^i_j) \to H^k_\mathfrak{m}(E_2^{-j,i+j}) \to H^{k+1}_\mathfrak{m}(\mathcal{M}^i_{j-1}) \to \cdots.$$

Lemma 6.5. *Adopt Setup 3.3, and given $i \in \mathbb{Z}$, let $\{\mathcal{M}^i_j\}$ be the filtration of $H^i_I(R)$ ensured by Corollary 4.10. If $j < n-i$, then $H^k_\mathfrak{m}(\mathcal{M}^i_j) = 0$ for all integers $k < n-i-j$.*

Proof. We proceed by induction on j. For the base case of $j = -1$, we have that $\mathcal{M}^i_{-1} = 0$, so $H^k_\mathfrak{m}(\mathcal{M}^j_{-1}) = 0$ for all integers k. Inductively, suppose that for some $j \geq 1$, $H^k_\mathfrak{m}(\mathcal{M}^i_{j-1}) = 0$ for all integers $k < n-i-(j-1)$.

Now assume that $k < n-i-j$, so that $H^k_\mathfrak{m}(\mathcal{M}^i_{j-1}) = H^{k+1}_\mathfrak{m}(\mathcal{M}^i_{j-1}) = 0$ by the inductive hypothesis since $k, k+1 < n-i-(j-1)$. The long exact sequence in Remark 6.4 implies that $H^k_\mathfrak{m}(\mathcal{M}^i_j) \cong H^k_\mathfrak{m}(E_2^{-j,i+j})$, and the statement now follows from Proposition 6.3. □

Lemma 6.6. *Adopt Setup 3.3, and given $i \in \mathbb{Z}$, let $\{\mathcal{M}^i_j\}$ be the filtration of $H^i_I(R)$ ensured by Corollary 4.10. Then for any $0 \leq j \leq n-i$, $H^k_\mathfrak{m}(\mathcal{M}^i_j) \cong H^k_\mathfrak{m}(H^i_I(R))$ for all $k > n-i-j$.*

Proof. We proceed by descending induction on j. For the base case of $j = n - i$, we have that $\mathcal{M}^i_j = H^i_I(R)$, so $H^k_{\mathfrak{m}}(\mathcal{M}^i_j) \cong H^k_{\mathfrak{m}}(H^i_I(R))$ for every $k \in \mathbb{Z}$.

Now suppose that $0 < j \le n - i$, and assume that $H^k_{\mathfrak{m}}(\mathcal{M}^i_j) \cong H^k_{\mathfrak{m}}(H^i_I(R))$ for all $k > n - i - j$. We aim to show that for all $k > n - i - (j-1)$, $H^k_{\mathfrak{m}}(\mathcal{M}^i_{j-1}) \cong H^k_{\mathfrak{m}}(H^i_I(R))$. Consider the long exact sequence in Remark 6.4,

$$\cdots \to H^{k-1}_{\mathfrak{m}}(E_2^{-j,i+j}) \to H^k_{\mathfrak{m}}(\mathcal{M}^i_{j-1}) \to H^k_{\mathfrak{m}}(\mathcal{M}^i_j) \to H^k_{\mathfrak{m}}(E_2^{-j,i+j}) \to \cdots .$$

If $k > n - i - (j-1)$, then $k - 1, k > n - i - j$, so $H^{k-1}_{\mathfrak{m}}(E_2^{-j,i+j}) = H^k_{\mathfrak{m}}(E_2^{-j,i+j}) = 0$ by Proposition 6.3, forcing $H^k_{\mathfrak{m}}(\mathcal{M}^i_{j-1})$ to be isomorphic to $H^k_{\mathfrak{m}}(\mathcal{M}^i_j)$, which in turn is isomorphic to $H^k_{\mathfrak{m}}(H^i_I(R))$ by the inductive hypothesis since k certainly exceeds $n - i - j$. □

Corollary 6.7. *Adopt Setup 3.3, and given $i \in \mathbb{Z}$, let $\{\mathcal{M}^i_j\}$ be the filtration of $H^i_I(R)$ ensured by Corollary 4.10. Then there exists an exact sequence of the form*

$$0 \to H^{n-i-j}_{\mathfrak{m}}(\mathcal{M}^i_j) \to H^{n-i-j}_{\mathfrak{m}}(E_2^{-j,i+j}) \to H^{n-i-j+1}_{\mathfrak{m}}(\mathcal{M}^i_{j-1}) \to H^{n-i-j+1}_{\mathfrak{m}}(H^i_I(R)) \to 0.$$

Proof. This follows from the long exact sequence from Remark 6.4, combined with our previous results: $H^{n-i-j}_{\mathfrak{m}}(\mathcal{M}^i_{j-1}) = 0$ by Lemma 6.5, and $H^{n-i-j+1}_{\mathfrak{m}}(E_2^{-j,i+j}) = 0$ by Proposition 6.3, while $H^{n-i-i+1}_{\mathfrak{m}}(\mathcal{M}^i_j) \cong H^{n-i-j+1}_{\mathfrak{m}}(H^i_I(R))$ by Lemma 6.6. □

Remark 6.8. Observe that if every minimal prime of a ring S is contained in some prime ideal Q of S, then for every $t \in \mathbb{Z}$, $\Lambda_t(S)$ and $\Lambda_t(S_Q)$ are canonically isomorphic simplicial complexes. Indeed, a simplex $\{P_1, \ldots, P_i\}$ of $\Lambda_t(S)$ corresponds to $\{P_1 S_Q, \ldots, P_i S_Q\} \in \Lambda_t(S_Q)$.

Lemma 6.9. *Adopt Setup 3.3, and for $j \in \mathbb{Z}$, let $\{\mathcal{M}^{n-j}_k\}_{k=-1}^{j}$ be the filtration of $H^{n-j}_I(R)$ ensured by Corollary 4.10. Moreover, let $A = R_{\mathfrak{m}}$, and $T = A/IA$. Then given $0 \le k \le j$, $H^k_{\mathfrak{m}}(\mathcal{M}^{n-j}_{j-k}) \cong E_A(\Bbbk)^{\oplus a}$, where*

$$a = (-1)^k \sum_{i=0}^{k} (-1)^i \lambda_{ij}(T) + (-1)^{k-1} \sum_{i=0}^{k-1} (-1)^i h_{j-i}(\Lambda_{d-i}(T), \Lambda_{d-i-1}(T); \Bbbk).$$

Proof. We proceed by induction on k. For $k = 0$, we have that $H^0_{\mathfrak{m}}(\mathcal{M}^{n-j}_j) = H^0_{\mathfrak{m}}(H^{n-j}_I(R))$, which by Remark 6.1 is isomorphic to the direct sum of $\lambda_{0j}(T)$ copies of $E_A(\Bbbk)$, aligning with the claim.

Suppose that for some $0 \le k < j$, $H^k_{\mathfrak{m}}(\mathcal{M}^{n-j}_{j-k}) \cong E_A(\Bbbk)^{\oplus a}$, with a as in the statement. The exact sequence guaranteed by Corollary 6.7, if $n - j$ takes the role of i and $j - k$ takes the role of j, has the form

$$0 \to H^k_{\mathfrak{m}}(\mathcal{M}^{n-j}_{j-k}) \to H^k_{\mathfrak{m}}(E_2^{-j+k,n-k}) \xrightarrow{\varphi} H^{k+1}_{\mathfrak{m}}(\mathcal{M}^{n-j}_{j-k-1}) \to H^{k+1}_{\mathfrak{m}}(H^{n-j}_I(R)) \to 0. \tag{6.1}$$

Observe that by the inductive hypothesis, the induced short exact sequence

$$0 \to H^k_{\mathfrak{m}}(\mathcal{M}^{n-j}_{j-k}) \to H^k_{\mathfrak{m}}(E_2^{-j+k,n-k}) \xrightarrow{\varphi} \mathrm{Im}(\varphi) \to 0$$

splits, and $H^k_{\mathfrak{m}}(E_2^{-j+k,n-k}) \cong E_A(\Bbbk)^{\oplus a} \oplus \mathrm{Im}(\varphi)$. By Proposition 6.3, $H^k_{\mathfrak{m}}(E_2^{-j+k,n-k}) \cong E_A(\Bbbk)^{\oplus b}$, where $b = h_{j-k}(\Lambda_{d-k}(T), \Lambda_{d-k-1}(T); \Bbbk)$ since $d = n-h$, noting Remark 6.8. Hence $\mathrm{Im}(\varphi)$ is a direct summand of an injective module, so is itself injective and is isomorphic to a direct sum of finitely many copies of $E_A(\Bbbk)$.

Then the short exact sequence

$$0 \to \operatorname{Im}(\varphi) \to H^{k+1}_{\mathfrak{m}}(\mathcal{M}^{n-j}_{j-k-1}) \xrightarrow{\psi} H^{k+1}_{\mathfrak{m}}(H^{n-j}_I(R)) \to 0$$

also induced by (6.1) splits, and since $H^{k+1}_{\mathfrak{m}}(H^{n-j}_I(R))$ is the direct sum of $\lambda_{k+1,j}(T)$ copies of $E_A(\Bbbk)$ by Remark 6.1, we conclude that $H^{k+1}_{\mathfrak{m}}(\mathcal{M}^{n-j}_{j-k-1}) \cong E_A(\Bbbk)^{\oplus c}$ for some integer $c \geq 0$. In particular, (6.1) is isomorphic to the exact sequence

$$0 \to E_A(\Bbbk)^{\oplus a} \to E_A(\Bbbk)^{\oplus b} \to E_A(\Bbbk)^{\oplus c} \to E_A(\Bbbk)^{\oplus \lambda_{k+1,j}(T)} \to 0.$$

Applying the Matlis duality functor $\operatorname{Hom}_A(-, E_A(\Bbbk))$, we obtain an exact sequence

$$0 \to \widehat{A}^{\oplus \lambda_{k+1,j}(T)} \to \widehat{A}^{\oplus c} \to \widehat{A}^{\oplus b} \to \widehat{A}^{\oplus a} \to 0,$$

where $\widehat{A}$ denotes the completion of A at $\mathfrak{m}A$. Taking ranks, we find that $c = \lambda_{k+1,j}(T)+b-a$, i.e., c is the sum of

$$\lambda_{k+1,j}(T) - (-1)^k \sum_{i=0}^{k} (-1)^i \lambda_{ij}(T) = (-1)^{k+1} \sum_{i=0}^{k+1} (-1)^i \lambda_{ij}(T)$$

and $h_{j-k}(\Lambda_{d-k}(T), \Lambda_{d-k-1}(T)) - (-1)^{k-1} \sum\limits_{i=0}^{k-1} (-1)^i h_{j-i}(\Lambda_{d-i}(S)\Lambda_{d-i-1}(T); \Bbbk)$, i.e.,

$$(-1)^k \sum_{i=0}^{k} (-1)^i h_{j-i}(\Lambda_{d-i}(T), \Lambda_{d-i-1}(T); \Bbbk)$$

since $(-1)^{2k+2} = (-1)^{2k} = 1$, completing the inductive step. □

Theorem 6.10. *Take S as in Setup 3.3, and let T be its localization at its homogeneous maximal ideal. Then for any integer $0 \leq j \leq d$,*

$$\sum_{i=0}^{j} (-1)^i \lambda_{ij}(T) = \sum_{i=0}^{j} (-1)^i h_{j-i}(\Lambda_{d-i}(T), \Lambda_{d-i-1}(T); \Bbbk)$$

Proof. The statement of Lemma 6.9 for $k = j$ says that for R and $\mathfrak{m}$ from Setup 3.3, if

$$a = (-1)^j \sum_{i=0}^{j} (-1)^i \lambda_{ij}(T) + (-1)^{j-1} \sum_{i=0}^{j-1} (-1)^i h_{j-i}(\Lambda_{d-i}(T), \Lambda_{d-i-1}(T); \Bbbk)$$

and $A = R_{\mathfrak{m}}$, then $H^j_{\mathfrak{m}}(\mathcal{M}^{n-j}_0) \cong E_A(\Bbbk)^{\oplus a}$. On the other hand, if $n-j$ takes the role of i and 0 takes the role of j, the exact sequence guaranteed by Corollary 6.7 has the form

$$0 \to H^j_{\mathfrak{m}}(\mathcal{M}^{n-j}_0) \to H^j_{\mathfrak{m}}(E_2^{0,n-j}) \xrightarrow{\varphi} H^{j+1}_{\mathfrak{m}}(\mathcal{M}^{n-j}_{-1}) \to H^{j+1}_{\mathfrak{m}}(H^{n-j}_I(R)) \to 0.$$

Now, $\mathcal{M}^{n-j}_{-1} = 0$ by definition, so $H^j_{\mathfrak{m}}(\mathcal{M}^{n-j}_0) \cong H^j_{\mathfrak{m}}(E_2^{0,n-j})$, which by Proposition 6.3 and Remark 6.8 is isomorphic to the direct sum of $h_0(\Lambda_{d-j}(T), \Lambda_{d-j-1}(T); \Bbbk)$ copies of $E_A(\Bbbk)$ since $n-h = d$. Hence $a = h_0(\Lambda_{d-j}(T), \Lambda_{d-j-1}(T); \Bbbk)$, and we obtain the claimed formula by multiplying both sides of this equation by $(-1)^{j-1}$, and then subtracting $\sum\limits_{i=0}^{j} (-1)^i \lambda_{ij}(T)$. □

Remark 6.11. If T is a d-dimensional equidimensional local ring containing a field, then as shown in [Lyu93, Property (4.4iii)], $\lambda_{ii}(T) = 0$ for all integers $0 \leq i \leq d-1$. Moreover, for all $t \geq 0$, $C_0(\Lambda_t(T); \Bbbk)$ is a $\Bbbk$-vector space with basis indexed by the minimal primes of I. Hence for every $0 \leq i \leq d-1$, $C_0(\Lambda_{d-i}(T), \Lambda_{d-i-1}(T); \Bbbk) = C_0(\Lambda_{d-i}(T); \Bbbk)/C_0(\Lambda_{d-i-1}(T); \Bbbk) = 0$, so that $h_0(\Lambda_{d-i}(T), \Lambda_{d-i-1}(T); \Bbbk) = 0$. Hence when T is equidimensional, Theorem 6.10 can be restated as

$$\sum_{i=0}^{j-1} (-1)^i \lambda_{ij}(T) = \sum_{i=0}^{j-1} (-1)^i h_{j-i}(\Lambda_{d-i}(T), \Lambda_{d-i-1}(T); \Bbbk).$$

Remark 6.12. Suppose that I is an ideal of an n-dimensional regular local ring $(A, \mathfrak{m}, \Bbbk)$ containing a field. By the Hartshorne-Lichtenbaum vanishing theorem [Har68, Theorem 3.1], $H_I^n(A) \neq 0$ if and only if $\dim(A/I) = 0$. Moreover, if $\dim(A/I) = 0$, then the spectral sequence $E_2^{p,q} = H_{\mathfrak{m}}^p(H_I^q(A)) \underset{p}{\Longrightarrow} H_{\mathfrak{m}}^{p+q}(A)$ [Har67, Proposition 1.4] has only one nonzero row. Thus, it degenerates on the E_2-page, so that $H_I^n(A) = H_{\mathfrak{m}}^0(H_I^n(A))$. Noting Remark 6.1, we conclude that $\dim(A/I) = 0$ if and only if $\lambda_{00}(A/I) = 0$. When $j = 0$, Theorem 6.10 is a restatement of this fact for localizations T of central linear subspace arrangements. Indeed, with Remark 2.4, it says that

$$\lambda_{00}(T) = h_0(\Lambda_d(T), \Lambda_{d-1}(T); \Bbbk) = h_{-1}(\Lambda_{d-1}(T); \Bbbk),$$

which is zero if and only if $\dim(T) \neq 0$.

Remark 6.13. Suppose that I is an ideal of an n-dimensional regular local ring $(A, \mathfrak{m}, \Bbbk)$ containing a field, such that $\Bbbk$ is separably closed. Assume that for every minimal prime P of I, $\dim(A/P) \geq 2$ (cf. [Wal01, Proposition 3.1], [HL90, Theorem 2.9]). Then $H_I^{n-1}(A)$ is the direct sum of $k-1$ copies of $E_A(\Bbbk)$, where k is the number of connected components of $\operatorname{Spec}^\circ(A/I)$, which is well understood to equal the number of connected components of (the 1-skeleton of) $\Lambda_{d-1}(A/I)$.

Indeed, this follows by the Second Vanishing Theorem of local cohomology when the punctured spectrum is connected [Ogu73, Corollary 2.11], [PS73, Chapitre III, Corollaire 5.5], [HL90, Theorem 2.9]. Let $I = J_1 \cap \ldots \cap J_k$, where each J_i defines a connected component of $\operatorname{Spec}^\circ(A/I)$. Inductively assume that $H^{n-1}_{J_1 \cap \ldots \cap J_{k-1}}(A) \cong E_A(\Bbbk)^{\oplus k-2}$. The Mayer-Vietoris long exact sequence in local cohomology with respect to $J := J_1 \cap \ldots \cap J_{k-1}$ and J_k has the form

$$\cdots \to H^{n-1}_{J+J_k}(A) \to H^{n-1}_J(A) \oplus H^{n-1}_{J_k}(A) \to H^{n-1}_I(A) \to H^n_{J+J_k}(A) \to H^n_J(R) \oplus H^n_{J_k}(A) \to \cdots$$

Since neither J nor J_k is $\mathfrak{m}$-primary, while $J + J_k$ is, we obtain the short exact sequence

$$0 \to H^{n-1}_J(A) \to H^{n-1}_I(A) \to H^n_{\mathfrak{m}}(A) \to 0$$

by the Hartshorne-Lichtenbaum vanishing theorem [Har68, Theorem 3.1], which splits by the inductive hypothesis, and

$$H^{n-1}_I(A) \cong H^{n-1}_J(A) \oplus H^n_{\mathfrak{m}}(A) \cong E_R(\Bbbk)^{\oplus k-2} \oplus E_R(\Bbbk) = E_R(\Bbbk)^{\oplus k-1}.$$

Hence in this setting, $H^0_{\mathfrak{m}}(H^{n-1}_I(A)) = H^{n-1}_I(A)$ and $H^1_{\mathfrak{m}}(H^{n-1}_I(A)) = 0$, and noting Remark 6.1, $\lambda_{01}(A/I) = k-1$, while $\lambda_{11}(A/I) = 0$.

Theorem 6.14. *Under Setup 3.3, assume that* $\Bbbk$ *is separably closed and that* S *is equidimensional of dimension at least* 3*, and let* T *be the localization of* S *at its homogeneous maximal ideal. Then*

$$\lambda_{12}(T) = h_0(\Lambda_{d-2}(T);\Bbbk) - h_0(\Lambda_{d-1}(T);\Bbbk), \text{ and}$$
$$\lambda_{02}(T) = h_0(\Lambda_{d-2}(T);\Bbbk) - h_0(\Lambda_{d-1}(T);\Bbbk) + h_1(\Lambda_{d-1}(T);\Bbbk) - h_1(\Lambda_{d-1}(T),\Lambda_{d-2}(T);\Bbbk).$$

Proof. By Remark 6.11 with $j = 2$ and Remark 2.4, we have that

$$\lambda_{02}(T) - \lambda_{12}(T) = h_1(\Lambda_{d-1}(T);\Bbbk) - h_1(\Lambda_{d-1}(T),\Lambda_{d-2}(T);\Bbbk). \tag{6.2}$$

By [NSW19, Theorem 5.4], $\lambda_{12}(T)$, which agrees with the $\lambda_{12}(\widehat{T})$, where $\widehat{A}$ denotes the completion of A at its maximal ideal [Lyu93, Lemma 4.2], equals the difference between the number of connected components of the 1-skeleton of $\Lambda_{d-2}(\widehat{T})$ and of the 1-skeleton of $\Lambda_{d-1}(\widehat{T})$. Now, in our setting, $\Lambda_t(T)$ is canonically isomorphic to $\Lambda_t(\widehat{T})$ for any t. Moreover, the number of connected components of the 1-skeleton of a simplicial complex agrees with the number of connected components of the complex itself. Thus,

$$\lambda_{12}(T) = h_0(\Lambda_{d-2}(T);\Bbbk) - h_0(\Lambda_{d-1}(T);\Bbbk).$$

Adding this value to both sides of (6.2) now yields the desired statement. □

Example 6.15. Consider again Example 2.2, assume that $\Bbbk$ is separably closed, and let T be the localization of S at its homogeneous maximal ideal. Recall that the nonzero values $h_k(\Lambda_t(S),\Lambda_{t-1}(S);\Bbbk)$, which agree with $h_k(\Lambda_t(T),\Lambda_{t-1}(T);\Bbbk)$ by Remark 6.8, are recorded in Example 5.3. Then $\lambda_{01}(T) = \widetilde{h}_0(\Lambda_{d-1}(T);\Bbbk) = 0$ by Remark 6.13, and Theorem 6.14 tells us that

$$\lambda_{12}(T) = 1 - 1 = 0 \quad\text{and}\quad \lambda_{02}(T) = 1 - 1 + 0 - 0 = 0.$$

Since T is equidimensional, Remark 6.11 further tells us that

$$\lambda_{03}(T) - \lambda_{13}(T) + \lambda_{23}(T) = 0 - 2 + 3 - 1 = 0, \text{ and}$$
$$\lambda_{24}(T) - \lambda_{34}(T) + \lambda_{44}(T) = 0 - 0 + 0 - 2 + 4 = 2$$

since $\lambda_{04}(T) = \lambda_{14}(T) = 0$ by analyzing the spectral sequence $E_2^{p,q} = H^p_{\mathfrak{m}}(H^q_I(R_{\mathfrak{m}})) \underset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{m}}(R_{\mathfrak{m}})$ [Har67, Proposition 1.4]. Indeed, all differentials to and from $E_r^{0,n-4}$ and $E_r^{1,n-4}$, for $r \geq 2$, must be zero.

In the setting of Theorem 6.14 when $d = 3$, every Lyubeznik number depends only on codimension complexes.

Corollary 6.16. *Under Setup 3.3, assume that* $\Bbbk$ *is separably closed,* S *is equidimensional of dimension* 3*, and* T *is the localization at its homogeneous maximal ideal. Then the only nonzero Lyubeznik numbers of* T *are the following, where* $h_\bullet(-) = h_\bullet(-;\Bbbk)$*:*

$$\lambda_{01}(T) = \widetilde{h}_0(\Lambda_2(T)),$$
$$\lambda_{02}(T) = \lambda_{23}(T) = h_0(\Lambda_1(T)) - h_0(\Lambda_2(T)) + h_1(\Lambda_2(T)) - h_1(\Lambda_2(T),\Lambda_1(T)),$$
$$\lambda_{12}(T) = h_0(\Lambda_2(T)) - h_0(\Lambda_1(T)), \text{ and}$$
$$\lambda_{33}(T) = h_0(\Lambda_1(T)).$$

In particular, the Lyubeznik numbers of T *depend only on* $\Lambda_1(T)$ *and* $\Lambda_2(T)$*.*

Proof. The value for $\lambda_{01}(T)$ follows from Remark 6.13, and $\lambda_{02}(T)$ and $\lambda_{12}(T)$ are given in Theorem 6.14. By [Lyu06, Theorem 1.3] and [Zha07, Main Theorem], $\lambda_{33}(T)$ is the number of connected components of the Hochster-Huneke graph of T, which is the same as the number of connected components of $\Lambda_1(T)$ since T is equidimensional, justifying the formula for $\lambda_{33}(T)$. Moreover, $\lambda_{00}(T) = 0$ by Remark 6.12.

Now, $\lambda_{ij}(T) = 0$ if $i > j$, and since T is equidimensional, $\lambda_{11}(T) = \lambda_{22}(T) = 0$ (see [Lyu93, Property (4.4iii)]). Therefore, it suffices to show that $\lambda_{03}(T) = \lambda_{13}(T) = 0$ and $\lambda_{23}(T) = \lambda_{02}(T)$. Let $A = R_{\mathfrak{m}}$, and consider the spectral sequence $E_2^{p,q} = H^p_{\mathfrak{m}}(H^q_I(A)) \underset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{m}}(A)$ [Har67, Proposition 1.4]. All differentials to and from $E_r^{0,n-3}$ and $E_r^{1,n-3}$, for $r \geq 2$, are zero because either their domain or target is the zero module, so both must be zero. Hence $\lambda_{03}(T) = \lambda_{13}(T) = 0$ by Remark 6.1. Moreover, the differential $E_2^{0,n-2} \to E_2^{2,n-3}$ must be an isomorphism since all maps to and from $E_r^{0,2}$ and $E_r^{1,3}$ for $r > 2$ are zero. Therefore, $\lambda_{23}(T) = \lambda_{02}(T)$, completing the proof. □

7. Examples

Example 7.1. Let $R = \Bbbk[x_1, \dots, x_5]$, where $\Bbbk$ is a field, and let I be the intersection of the ideals $I_1 = \langle x_1, x_2 \rangle$, $I_2 = \langle x_3, x_4 \rangle$, $I_3 = \langle x_1, x_5 \rangle$, and $I_4 = \langle x_3, x_5 \rangle$ of R. Moreover, for $1 \leq i \leq 4$, let P_i denote the image of I_i in $S = R/I$. Then S is 3-dimensional, and under this ordering of minimal primes, $\Lambda_1(S)$ and $\Lambda_2(S)$ appear in Figure 7.1.1. Hence, one can find that there are only four nonzero consecutive relative homology modules of the codimension complexes, $h_2(\Lambda_3(S), \Lambda_2(S); \Bbbk) = 1$, $h_1(\Lambda_2(S), \Lambda_1(S); \Bbbk) = 1$, $h_1(\Lambda_1(S), \Lambda_0(S); \Bbbk) = 3$, and $h_0(\Lambda_0(S); \Lambda_{-1}(S); \Bbbk) = 4$. Additionally, there are nonzero simplicial homology modules of the codimension complexes, among others we have $h_0(\Lambda_1(S); \Bbbk) = 1$, $\widetilde{h}_0(\Lambda_2(S); \Bbbk) = 0$, and $h_1(\Lambda_2(S); \Bbbk) = 1$.

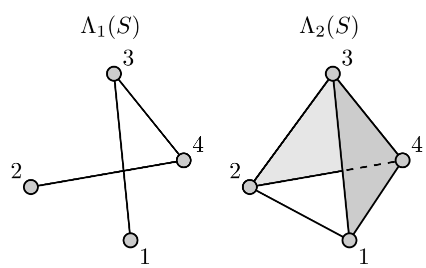

Figure 7.1.1. The codimension complexes of S from Example 7.1.

By Theorem 5.1, $H^i_I(R) = 0$ unless $i = 2$ or $i = 3$. According to Theorem 5.14, if $\Bbbk$ has characteristic zero, then $H^2_I(R)$ has $\mathcal{D}(R, \Bbbk)$-module length $4 + 3 = 7$, and $H^3_I(R)$ has length $1 + 1 = 2$.

What's more, if $\mathfrak{m}$ denotes the homogeneous maximal ideal of R, Corollary 4.10 guarantees the existence of short exact sequences with the following forms.

$$0 \to \bigoplus_{i=1}^4 H^2_{I_i}(R) \to H^2_I(R) \to H^3_{\langle x_1,x_2,x_5 \rangle}(R) \oplus H^3_{\langle x_3,x_4,x_5 \rangle}(R) \oplus H^3_{\langle x_1,x_3,x_5 \rangle}(R) \to 0$$
$$0 \to H^4_{\langle x_1,x_2,x_3,x_4 \rangle}(R) \to H^3_I(R) \to H^5_{\mathfrak{m}}(R) \to 0$$

If T is the localization of S at its homogeneous maximal ideal and $\Bbbk$ is separably closed, then Corollary 6.16 provides that $\lambda_{ij}(T) = 1$ for $i = j = 3$, and otherwise vanishes.

Example 7.2. Consider the braid arrangement defined by the ideal

$$I = \bigcap_{1 \le i < j \le 4} \langle x_i - x_j \rangle \subseteq R = \Bbbk[x_1, x_2, x_3, x_4]$$

where $\Bbbk$ is a field. Then $S = R/I$ is equidimensional of dimension three, with six minimal primes. The complex $\Lambda_1(S)$ has an edge between all pairs of the six vertices, and four 2-cells corresponding to the sum of all triples of the minimal primes of the form $\langle x_i - x_j \rangle$, $\langle x_j - x_k \rangle$, and $\langle x_i - x_k \rangle$ for i, j, k distinct. Moreover, both $\Lambda_2(S)$ and $\Lambda_3(S)$ are the full simplicial complex on six vertices. The nonzero values $h_k(\Lambda_t(S); \Bbbk)$ are

$$h_0(\Lambda_1(S); \Bbbk) = h_0(\Lambda_2(S); \Bbbk) = h_0(\Lambda_3(S); \Bbbk) = 1,$$

and the nonzero $h_k(\Lambda_t(S), \Lambda_{t-1}(S); \Bbbk)$ are

$$h_0(\Lambda_0(S), \Lambda_{-1}(S); \Bbbk) = h_2(\Lambda_2(S), \Lambda_1(S); \Bbbk) = 6 \text{ and } h_1(\Lambda_1(S), \Lambda_0(S); \Bbbk) = 11.$$

By Theorem 5.1, the only nonvanishing local cohomology module of R with support in I is $H^1_I(R)$. Moreover, if $\Bbbk$ has characteristic zero, then $H^1_I(R)$ has $\mathcal{D}(R, \Bbbk)$-module length $6 + 11 + 6 = 23$ according to Theorem 5.14.

In general, if $\Bbbk$ is separably closed and T is the localization of S at its homogeneous maximal ideal, then Theorem 6.14 tell us us that that $\lambda_{12}(T) = 1 - 1 = 0$ and $\lambda_{02}(T) = 1 - 1 + 0 - 0 = 0$ by Theorem 6.14, while Corollary 6.16 further shows that $\lambda_{33}(T) = 1$, and all other Lyubeznik numbers vanish.

Remark 7.3. Recall that Theorem 6.10 identifies two alternating sums of integers. Example 7.2 with $j = 3$ shows that the terms in the two sums need not correspond in any way. Indeed, here, the statement of Theorem 6.10 has the form

$$0 - 0 + 0 - 1 = 0 - 6 + 11 - 6.$$

Example 7.4. Let $R = \Bbbk[x_1, \ldots, x_5]$, where $\Bbbk$ is a field, and let I be the intersection of the ideals $I_1 = \langle x_1, x_4 \rangle$, $I_2 = \langle x_2, x_5 \rangle$, and $I_3 = \langle x_1, x_2, x_3 \rangle$ of R. Observe that $S = R/I$ has dimension three, and is not equidimensional. If, for $1 \le i \le 3$, P_i is the image of I_i in S, then the corresponding codimension complexes of S with respect to the ordering P_1, P_2, P_3 appear in Figure 7.4.1. The nonzero $h_k(\Lambda_t(S), \Lambda_{t-1}(S); \Bbbk)$ are $h_2(\Lambda_3(S), \Lambda_2(S); \Bbbk) = 1$, $h_1(\Lambda_2(S), \Lambda_1(S); \Bbbk) = 3$, $h_0(\Lambda_1(S), \Lambda_0(S); \Bbbk) = 1$, and $h_0(\Lambda_0(S); \Lambda_{-1}(S); \Bbbk) = 2$.

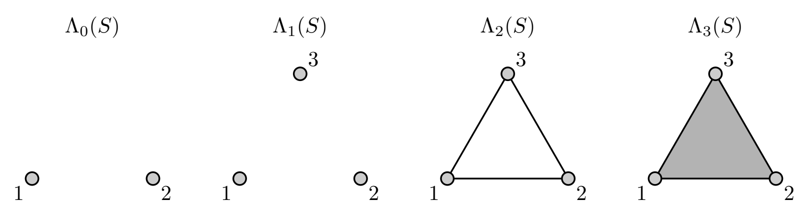

FIGURE 7.4.1. The simplicial complexes of S from Example 7.4.

By Theorem 5.1, the only nonzero local cohomology modules $H^i_I(R)$ occur at indices $i = 2$ and $i = 3$. According to Theorem 5.14, if $\Bbbk$ is of characteristic zero, then $H^2_I(R)$ has $\mathcal{D}(R, \Bbbk)$-module length 2, and $H^3_I(R)$ has length $1 + 3 + 1 = 5$.

What's more, if $\mathfrak{m}$ denotes the homogeneous maximal ideal of R, Corollary 4.10 guarantees that $H^2_I(R) \cong H^2_{\langle x_1, x_4 \rangle}(R) \oplus H^2_{\langle x_2, x_5 \rangle}(R)$. Additionally, it guarantees the existence of short exact sequences

$$0 \to M \to H^3_I(R) \to H^5_{\mathfrak{m}}(R) \to 0$$
$$0 \to H^3_{\langle x_1,x_2,x_3\rangle}(R) \to M \to H^4_{\langle x_1,x_2,x_4,x_5\rangle}(R) \oplus H^4_{\langle x_1,x_2,x_3,x_4\rangle}(R) \oplus H^4_{\langle x_1,x_2,x_3,x_5\rangle}(R) \to 0$$

where M denotes the R-submodule $\mathcal{M}^3_1$ of $H^3_I(R)$ as in Corollary 4.10.

Àlvarez Montaner and Vahidi study this example in the case that $\Bbbk = \mathbb{Q}$ in [ÀV14, Example 5.7], obtaining the same statement on (non-)vanishing of local cohomology as we do over arbitrary $\Bbbk$. Therein, they also find that the only nonzero Lyubeznik numbers of the localization T of S at $\langle x_1, \dots, x_5\rangle$ as $\lambda_{12}(T) = 1$ and $\lambda_{33}(T) = 2$. Interestingly, the formula for $\lambda_{12}(T)$ in Theorem 6.14 fails in this case since

$$h_0(\Lambda_1(T); \mathbb{Q}) - h_0(\Lambda_2(T); \mathbb{Q}) = 3 - 1 = 2.$$

Example 7.5. Consider the Stanley-Reisner ideal associated to a minimal triangulation of the real projective plane in $R = \Bbbk[x_1, \dots, x_6]$, i.e., I is the intersection of the following prime ideals:

$$\begin{array}{lllll} I_1 = \langle x_1,x_2,x_3\rangle & I_2 = \langle x_1,x_2,x_4\rangle & I_3 = \langle x_1,x_3,x_5\rangle & I_4 = \langle x_2,x_4,x_5\rangle & I_5 = \langle x_3,x_4,x_5\rangle \\ I_6 = \langle x_2,x_3,x_6\rangle & I_7 = \langle x_1,x_4,x_6\rangle & I_8 = \langle x_3,x_4,x_6\rangle & I_9 = \langle x_1,x_5,x_6\rangle & I_{10} = \langle x_2,x_5,x_6\rangle. \end{array}$$

Observe that $S = R/I$ is equidimensional of dimension 3.

The codimension complex $\Lambda_1(S)$ consists of fifteen edges, $\Lambda_2(S)$ contains forty-five edges, sixty 2-cells, thirty 3-cells, and six 4-cells, while $\Lambda_3(S)$ is the full simplicial complex on the ten vertices. Table 1 shows the nonzero $h_j(\Lambda_t(S), \Lambda_{t-1}(S); \Bbbk)$ when $\Bbbk = \mathbb{Q}$, and when $\Bbbk = \mathbb{F}_2$.

For $\Bbbk = \mathbb{Q}$, $\widetilde{h}_k(\Lambda_t(S); \mathbb{Q}) = 6$ when $k = t = 1$, and is zero otherwise. For $\Bbbk = \mathbb{F}_2$, the nonzero $\widetilde{h}_j(\Lambda_t(S); \mathbb{F}_2)$ are $h_1(\Lambda_1(S); \mathbb{F}_2) = 6$, $h_1(\Lambda_2(S); \mathbb{F}_2) = 1$, and $h_2(\Lambda_2(S); \mathbb{F}_2) = 6$.

	$\Bbbk = \mathbb{Q}$				$\Bbbk = \mathbb{F}_2$			
t \ j	0	1	2	3	0	1	2	3
0	10	0	0	0	10	0	0	0
1	0	15	0	0	0	15	0	0
2	0	0	6	0	0	0	6	0
3	0	0	0	0	0	0	1	1

TABLE 1. The dimensions of the relative homology $h_j(\Lambda_t(S), \Lambda_{t-1}(S); \Bbbk)$ with coefficients in two different fields $\Bbbk$ considered in Example 7.5.

By Theorem 5.1, $H^3_I(R)$ is the only nonzero local cohomology module with support in I when $\Bbbk = \mathbb{Q}$. However, $H^3_I(R)$ and $H^4_I(R)$ are the nonzero local cohomology modules with support in I when $\Bbbk = \mathbb{F}_2$. Hence, the cohomological dimension depends on the characteristic of $\Bbbk$. By Theorem 5.14, when $\Bbbk = \mathbb{Q}$, we have the length of $H^3_I(R)$ as a $\mathcal{D}(R, \Bbbk)$-module is $10 + 15 + 6 + 0 = 31$.

Let T denote the localization of S at the homogeneous maximal ideal. Using Corollary 6.16, we have that when $\Bbbk = \mathbb{Q}$, the nonzero Lyubeznik numbers are $\lambda_{33}(T) = 1$. Alternatively, when $\Bbbk = \mathbb{F}_2$, the nonzero Lyubeznik numbers are $\lambda_{02}(T) = \lambda_{23}(T) = \lambda_{33}(T) = 1$.

8. Depth and Cohomological Dimension for Squarefree Monomial Ideals

Suppose that an ideal I of a polynomial ring R over a field defines a central linear subspace arrangement. The minimal primes of I are all generated by variables if and only if I is a squarefree monomial ideal, or in other words, R/I is a Stanley-Reisner ring.

The goal of this section is to provide a new proof, in the squarefree case, of the strong relationship between cohomological dimension and depth for monomial ideals, originally proved by Lyubeznik [Lyu84, Theorem 1(iv)]. We apply, in combination, our characterization of the cohomological dimension from Corollary 5.2, and the description of depth for Stanley-Reisner rings recently established in [DDD+19].

Theorem 8.1. *If I is a squarefree monomial ideal of a polynomial ring R over a field, then*

$$\operatorname{cd}(R, I) = \dim(R) - \operatorname{depth}(R/I).$$

Before proving Theorem 8.1, we recall the notion of the "higher nerves" of a simplicial complex, introduced in [DDD+19, Definition 1.2], and then realize them as codimension complexes.

Definition 8.2. Fix an ordering $\tau_1, \tau_2, \ldots, \tau_m$ of the facets of a simplicial complex Δ of dimension $d-1$. Given an integer t, the t-th nerve complex of Δ is defined as the simplicial complex $N_t(\Delta) \subseteq 2^{[m]}$ whose faces are defined as follows: Given $\sigma \subseteq [m]$,

$$\sigma \in N_t(\Delta) \text{ if and only if } \Big|\bigcap_{i\in\sigma} \tau_i\Big| \geq t.$$

Observe that for $t > d$, $N_t(\Delta) = \{\varnothing\}$, while if $t \leq 0$, $N_t(\Delta)$ is the full simplex on $[m]$. Moreover, this family indeed includes the *nerve* of Δ as $N_1(\Delta)$, since the condition that $|\bigcap_{i\in\sigma} \tau_i| \geq 1$ is equivalent to that defining the nerve, i.e., $\bigcap_{i\in\sigma} \tau_i = \varnothing$. By Borsuk's Nerve Theorem, a simplicial complex and its nerve have the same homotopy type [Bor48, Section 9, Corollary 2].

Now we turn to relating the higher nerves to the codimension complexes. Toward this, fix a field $\Bbbk$. Recall that given a simplicial complex Δ on vertex set $[n]$ for some $n \in \mathbb{N}$, the corresponding Stanley-Reisner ideal I_Δ of $R = \Bbbk[x_1, \ldots, x_n]$ is the squarefree monomial ideal generated by all monomials $x_{i_1}x_{i_2}\cdots x_{i_j}$ for which $\sigma = \{i_1, i_2, \ldots, i_j\} \notin \Delta$. The Stanley-Reisner ring of Δ is R/I_Δ. The correspondence $\Delta \mapsto I_\Delta$ is, in fact, a bijection between simplicial complexes on $[n]$ and squarefree monomial ideals of R.

The Stanley-Reisner ideal $I_\Delta \subseteq R$ can be written as the intersection of the prime monomial ideals of the form $Q_\sigma = \langle x_i \mid i \notin \sigma\rangle$, over all $\sigma \in \Delta$ (see, e.g., [MS05, Theorem 1.7]). If $\sigma \subseteq \tau$ are faces of Δ, then $Q_\tau \subseteq Q_\sigma$. Hence I is the intersection of the Q_σ for which σ is a facet of Δ, and this intersection is minimal. Therefore, the map $\tau \mapsto Q_\tau$ defines a bijection between the facets of Δ and the minimal primes of I_Δ. In particular, $\dim(R/Q_\tau) = |\tau|$, and $\dim(R/I_\Delta)$ is the maximum of $|\tau|$ over all the facets τ of Δ. Hence $\dim(R/I_\Delta) = \dim \Delta + 1$.

Proposition 8.3. *Suppose that Δ is a $d-1$-dimensional simplicial complex on vertex set $[n]$ for some positive integer n. Let I_Δ be the corresponding Stanley-Reisner ideal of $R = \Bbbk[x_1, \ldots, x_n]$, where $\Bbbk$ is a field. Then for every integer $t \leq d+1$, $N_t(\Delta)$ and $\Lambda_{d-t}(R/I_\Delta)$ are isomorphic simplicial complexes.*

Proof. The discussion preceding this proposition describes a bijection between the facets of Δ and the minimal primes of I_Δ, under which a facet $\tau \in \Delta$ corresponds to the ideal

$Q_\tau = \langle x_i \mid i \notin \tau \rangle$. Thus, an ordering $\tau_1, \ldots, \tau_\ell$ of the facets of Δ induces an ordering $P_1, \ldots, P_\ell$ of the minimal primes of $S = R/I_\Delta$, where P_i is the image of Q_{τ_i} in S.

Observe that $\dim(S) = \dim(\Delta) + 1 = d$. Hence if $t = d + 1$, then $N_{d+1}(\Delta) = \{\varnothing\} = \Lambda_{-1}(S)$, and the statement holds. Suppose instead that $t \leq d$. Then the vertices of $N_t(\Delta)$ are the values $1 \leq i \leq \ell$ for which $|\tau_i| \geq t$, i.e., those for which $\dim(R/Q_{\tau_i}) \geq t$. Since $S/P_i \cong R/Q_{\tau_i}$, this is exactly the condition for i to be a vertex of $\Lambda_{d-t}(S)$. Hence, there is a canonical bijection between the vertex set of $N_t(\Delta)$, and that of $\Lambda_{d-t}(S)$.

It remains to verify that this bijection of the vertices preserves faces. If $t \leq 0$, this holds since $N_t(\Delta) = \Lambda_{d-t}(S) = 2^{[\ell]}$. Suppose instead that $1 \leq t \leq d$. Given $\sigma \subseteq [\ell]$, then $\sigma \in N_t(\Delta)$ if and only if the intersection of the τ_i, over all $i \in \sigma$, contains $\{i_1, \ldots, i_t\}$ for some $1 \leq i_1 < \cdots < i_t \leq \ell$. Equivalently, for every $i \in \sigma$, $\{i_1, \ldots, i_t\} \subseteq \tau_i$. Now, this happens if and only if for every $i \in \sigma$, Q_{τ_i} contains none of $x_{i_1}, \ldots, x_{i_t}$, i.e., $Q_{\tau_1} + \cdots + Q_{\tau_\ell} \subseteq R$ contains none of these indeterminates. We conclude that $\sigma \in N_t(\Delta)$ if and only if $\mathrm{height}_R(\mathbb{J}(\sigma)) \leq n - t$, which holds if and only if $\sigma \in \Lambda_{d-t}(S)$ by Remark 2.3. $\square$

Given a simplicial complex Δ on $[n]$, [DDD+19, Theorem 5.2] states that the corresponding Stanley-Reisner ideal I_Δ of $R = \Bbbk[x_1, \ldots, x_n]$, where $\Bbbk$ is a field, satisfies

$$\mathrm{depth}(R/I_\Delta) = \min\{i + j \mid \widetilde{H}_j(N_i(\Delta); \Bbbk) \neq 0\}.$$

Though this result is originally stated with an infimum replacing the above minimum, we note that the minimum exists. Since $N_{d+1}(\Delta) = \{\varnothing\}$, we have that $H_{-1}(N_{d+1}(\Delta); \Bbbk) \neq 0$, so the set under consideration is nonempty. Moreover, $\widetilde{H}_j(N_t(\Delta); \Bbbk) = 0$ if either $j < -1$ or if $t \leq 0$, so the set is bounded below.

The use of $N_{d+1}(\Delta)$ is necessary here: For instance, if Δ is the full simplicial complex on $[n]$, then I_Δ is the zero ideal, and $R/I_\Delta = R = \Bbbk[x_1, \ldots, x_n]$ has depth n. Hence $N_t(\Delta)$ consists solely of one vertex for all $i \leq d$, so that all their reduced homology vanishes. However, $H_{-1}(N_{d+1}(\Delta); \Bbbk) \neq 0$, confirming that R has depth n.

We are now ready to prove Theorem 8.1.

Proof of Theorem 8.1. For $\Bbbk$ a field, let $R = \Bbbk[x_1, \ldots, x_n]$, and let Δ be the simplicial complex determined by the Stanley-Reisner correspondence for which $I = I_\Delta$. Then

$$\mathrm{cd}(R, I) = n - \min\{i + j \mid i, j \in \mathbb{Z} \text{ and } \widetilde{H}_j(\Lambda_{d-i}(S); \Bbbk) \neq 0\}$$

by Corollary 5.2. Moreover, $\widetilde{H}_j(\Lambda_{d-i}(S); \Bbbk)$ can be replaced by $\widetilde{H}_j(N_i(\Delta); \Bbbk)$ by Proposition 8.3: Though $N_i(\Delta)$ is defined for any $i \in \mathbb{Z}$, the minimum occurs for $i \leq d + 1$ since $N_t(\Delta) = \{\varnothing\}$ for all $t \geq d+1$, and $i \leq d+1$ if and only if $d-i \geq -1$, i.e., $\Lambda_{d-i}(S)$ is defined. Finally, after this replacement, the minimum value in this formula equals $\mathrm{depth}(R/I)$ by [DDD+19, Theorem 5.2], yielding the claimed formula. $\square$

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